FACTORIZATION OF DISCRETE DYNAMICAL SYSTEMS

AUTOMATA & WAN 2021
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DDS and Complex behaviours

Equations over DDS

Abstractions over DDS

- Over the cardinality of the set of states
- Over the asymptotic behaviour
- Over the transient behaviour

The Cyclic Abstraction

- The basic case (EnumSOBFID Problem)
- Contractions Steps
- W-th roots

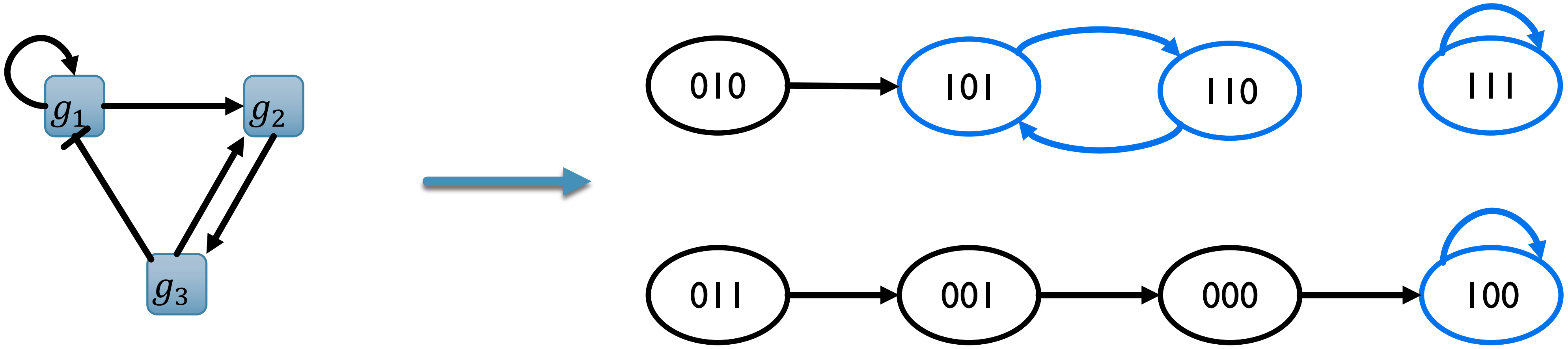
Conclusions and Perspectives

DISCRETE DYNAMICAL SYSTEMS

- A DDS is a pair $< \mathrm{X}, f >$, where X is the **set of states** and f is the **next state map**.

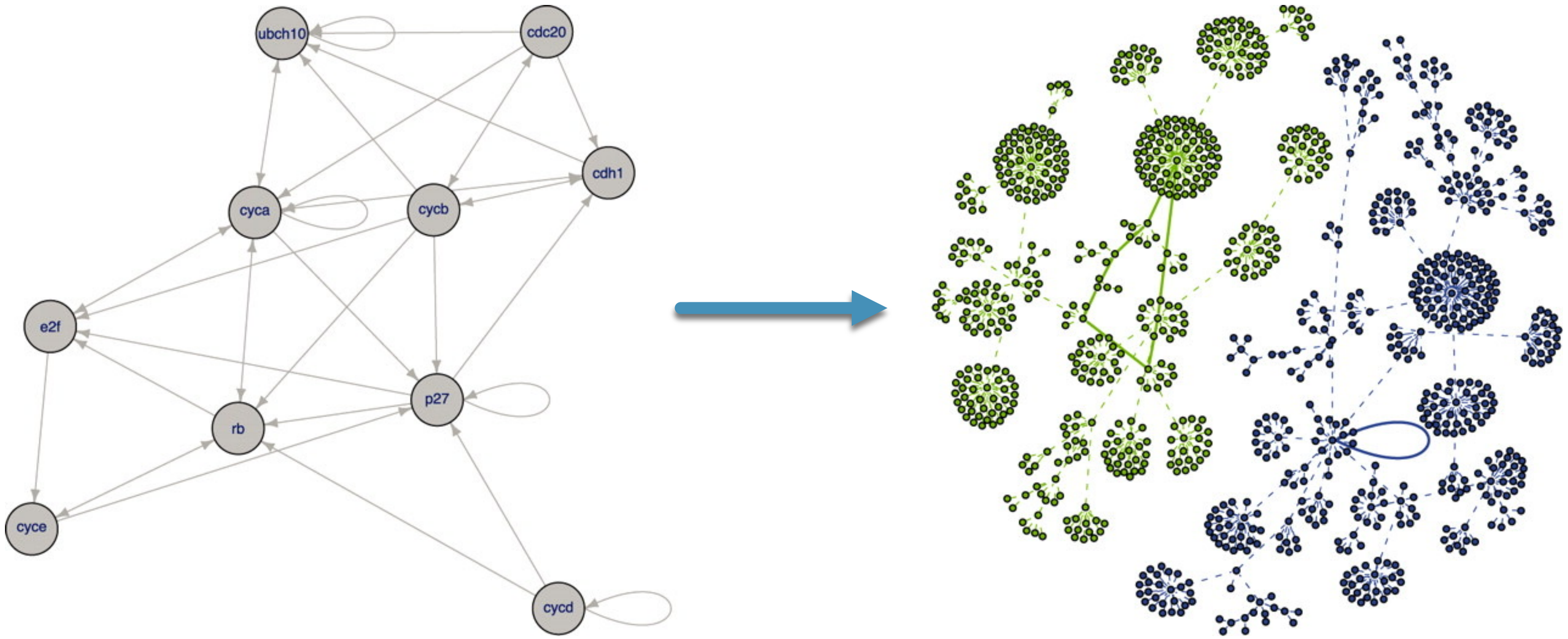
$$f : \mathrm{X} \to \mathrm{X}$$
$$x \longmapsto f(x)$$

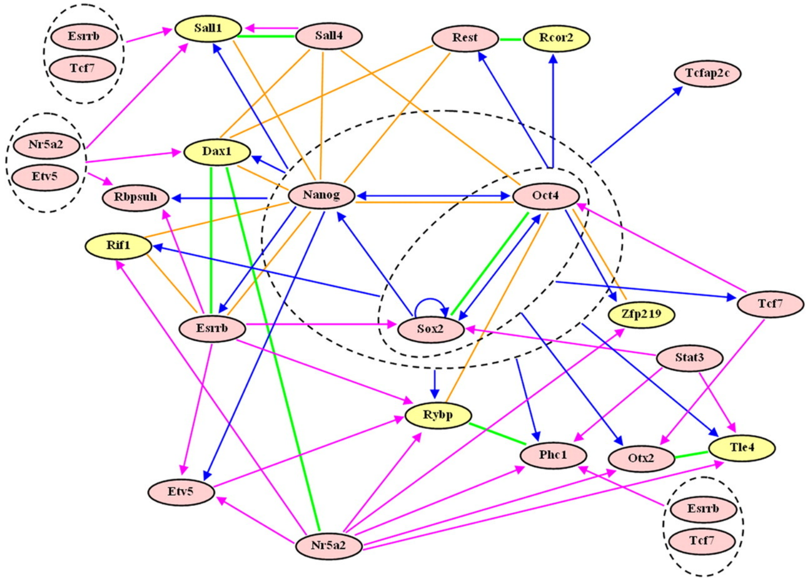
- Any DDS can be identified with its **dynamics graph** $\mathrm{G} \equiv (V, E)$ where $\mathrm{V} = \mathrm{X}$ and $\mathrm{E} = \{(\alpha, \beta) \in V \times V, f(\alpha) = \beta\}$.

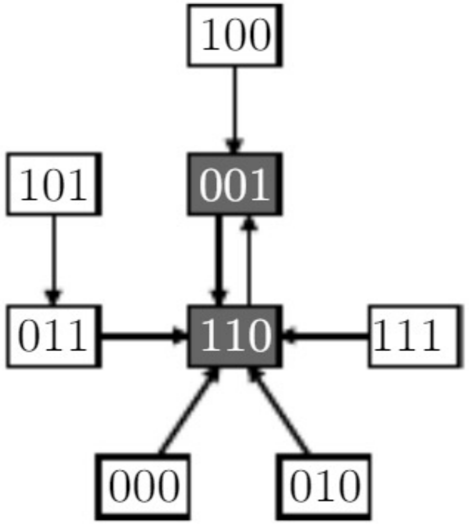


DISCRETE DYNAMICAL SYSTEMS

- **Complex dynamics** in Discrete Dynamical Systems
- Simulations or Verification are in general unfeasible

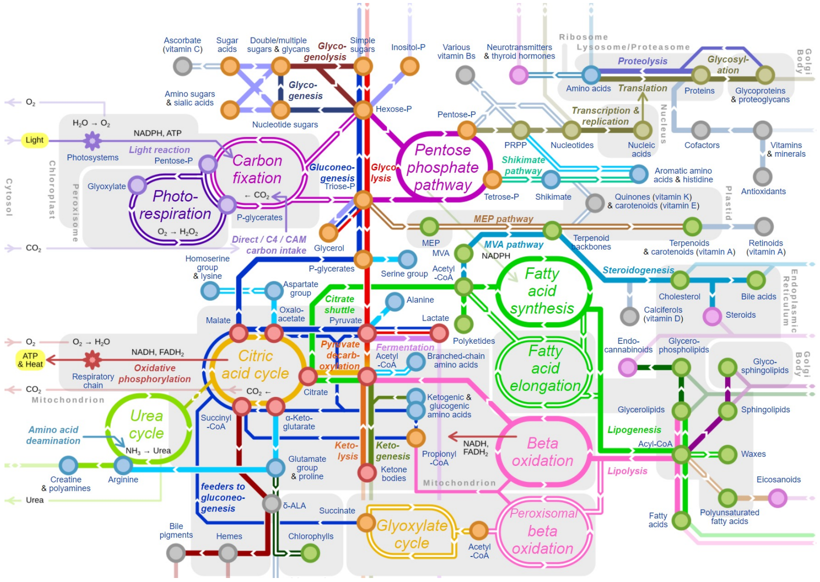






EQUATIONS OVER DDS

- Polynomial equations to translate hypothesis on complex dynamics
- Solutions provide the validation of the hypothesis



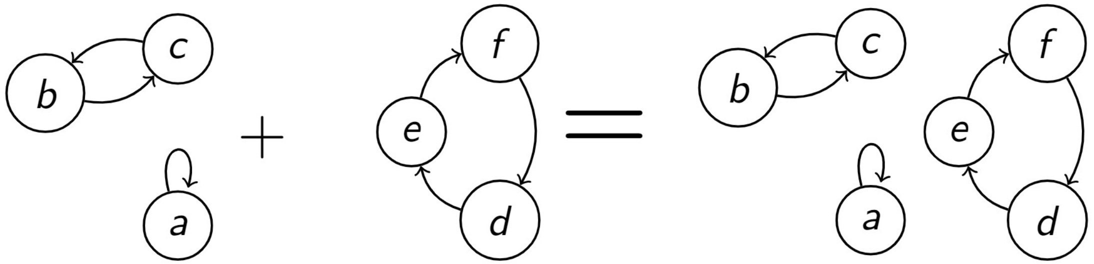
A COMMUTATIVE SEMIRING

SUM

$$< \mathbf{X}, \boldsymbol{f} > + < \boldsymbol{\Upsilon}, \boldsymbol{g} > = < \mathbf{X} \sqcup \boldsymbol{\Upsilon}, \boldsymbol{f} \sqcup \boldsymbol{g} >$$

$$\boldsymbol{f} \sqcup \boldsymbol{g} : \mathbf{X} \sqcup \boldsymbol{\Upsilon} \to \mathbf{X} \sqcup \boldsymbol{\Upsilon}$$

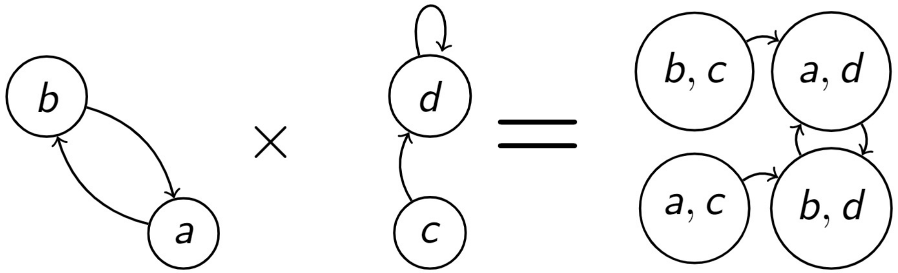
$$\forall\, (\boldsymbol{\alpha}, \boldsymbol{i}) \in \mathbf{X} \sqcup \boldsymbol{\Upsilon} \quad (\boldsymbol{f} \sqcup \boldsymbol{g})(\boldsymbol{\alpha}, \boldsymbol{i}) = \begin{cases} (\boldsymbol{f}(\boldsymbol{\alpha}), \boldsymbol{i}) & \boldsymbol{if} \quad \boldsymbol{\alpha} \in \mathbf{X} \wedge \boldsymbol{i} = \mathbf{0} \\ (\boldsymbol{g}(\boldsymbol{\alpha}), \boldsymbol{i}) & \boldsymbol{if} \quad \boldsymbol{\alpha} \in \boldsymbol{\Upsilon} \wedge \boldsymbol{i} = \mathbf{1} \end{cases}$$



PRODUCT

$$< \mathbf{X}, \boldsymbol{f} > \times < \boldsymbol{\Upsilon}, \boldsymbol{g} > = < \mathbf{X} \times \boldsymbol{\Upsilon}, \boldsymbol{f} \times \boldsymbol{g} >$$

$$(\boldsymbol{f} \times \boldsymbol{g})(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{f}(\boldsymbol{\alpha}), \boldsymbol{g}(\boldsymbol{\beta}))$$



HYPOTHESIS VALIDATION

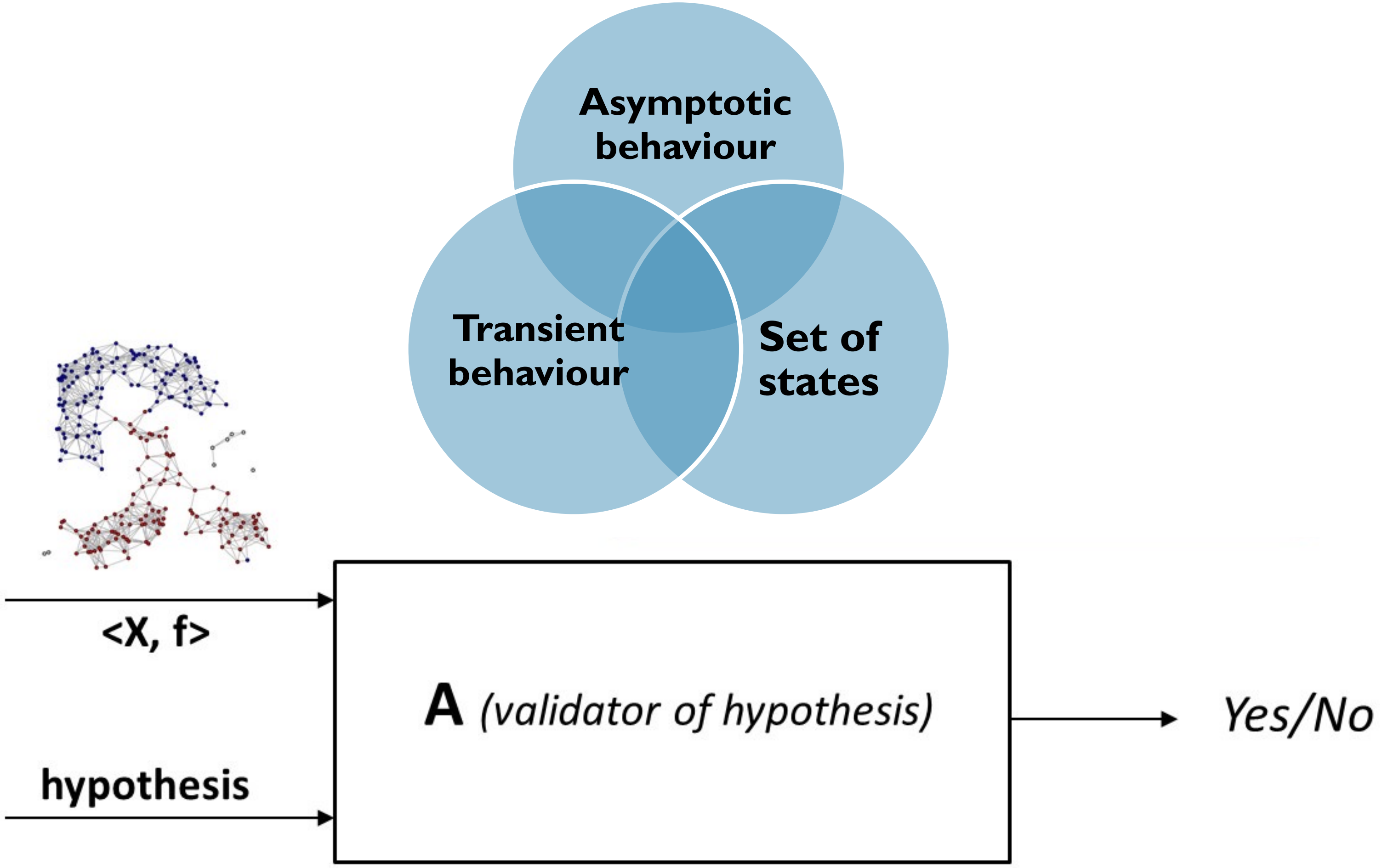
$$\boldsymbol{a_1 \cdot x_1^{w_1} + a_2 \cdot x_2^{w_2} + \ \ldots + a_s \cdot x_s^{w_s} = C}$$

The equation admits a solution → the hypothesis is verified

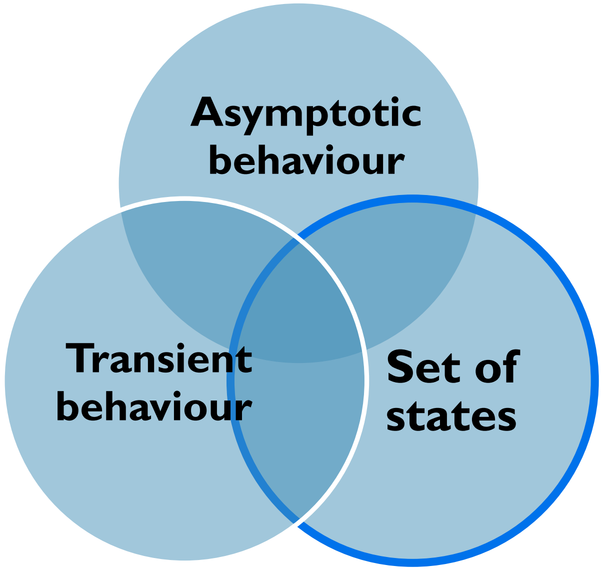
(Dennunzio, Dorigatti, Formenti, Manzoni and Porreca, 2018)
It is proved that:

- the set of DDS equipped with these operations of sum and product is a **commutative semiring**
- the problem of finding a solution for $P(x_1, \ldots, x_s) = Q(x_1, \ldots, x_s)$ is **undecidable**
- With a constant term, the complexity is beyond NP

THE MAIN IDEA



ABSTRACTION OVER THE CARDINALITY OF STATES



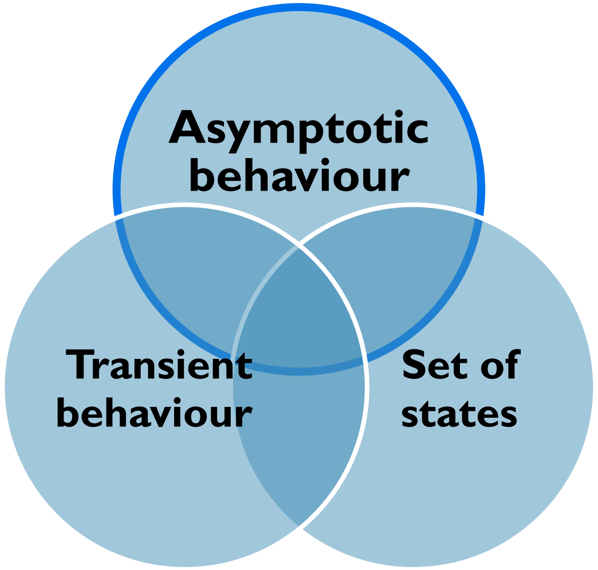
$$\langle X_1, f_1 \rangle x_1^{w_1} + \langle X_2, f_2 \rangle x_2^{w_2} + \cdots + \langle X_s, f_s \rangle x_s^{w_s} = \langle \Upsilon, g \rangle$$

$$\downarrow$$

$$|X_1| \bar{x}_1^{w_1} + |X_2| \bar{x}_2^{w_2} + \cdots + |X_S| \bar{x}_S^{w_s} = |\Upsilon|$$

- In the this abstraction, coefficients, variables and known term are natural numbers.
- $\bar{x}_i$ is the cardinality of the set of states in the variable.

ABSTRACTION OVER THE ASYMPTOTIC BEHAVIOUR



$$\langle \mathrm{X}_1, f_1 \rangle x_1^{w_1} + \langle \mathrm{X}_2, f_2 \rangle x_2^{w_2} + \cdots + \langle \mathrm{X}_s, f_s \rangle x_s^{w_s} = \langle \Upsilon, g \rangle$$

↓

$$\left(\bigoplus_{i=1}^{K_1} C_{p_{1i}}^{n_{1i}} \odot \dot{x}_1^{w_1} \right) \oplus \left(\bigoplus_{i=1}^{K_2} C_{p_{2i}}^{n_{2i}} \odot \dot{x}_2^{w_2} \right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{K_s} C_{p_{si}}^{n_{si}} \odot \dot{x}_s^{w_s} \right) = \bigoplus_{j=1}^{m} C_{p_j}^{n_j}$$

In the decidable abstraction, coefficients, variables and known term are cycles.

THE NOTATION

Given $A \equiv \langle X, f \rangle$ and Π its set of periodic points, we denote $\dot{A}$ the DDS induced by Π.

$$\dot{A} = \bigoplus_{i=1}^{k} C_{p_i}^{n_i}$$

An example…

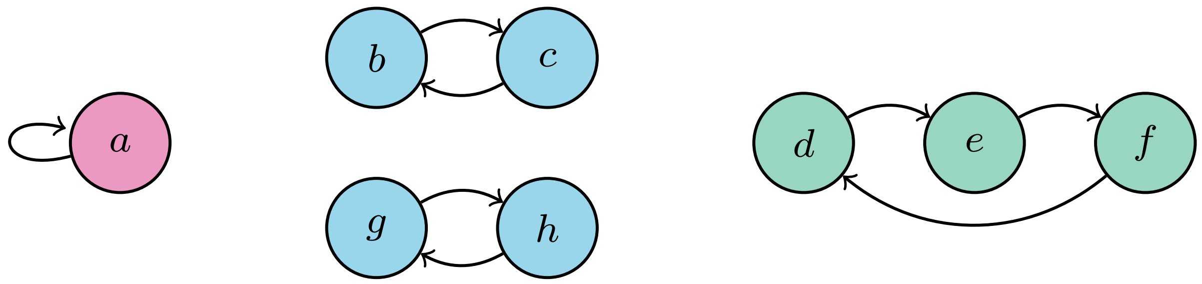


Figure: $(C_1^1 \oplus C_2^2 \oplus C_3^1)$.

OPERATIONS OVER CYCLES

$\dot{A} \oplus \dot{B}$

$$\bigoplus_{i=1}^{K_A} C_{p_{Ai}}^{n_{Ai}} \oplus \bigoplus_{j=1}^{K_B} C_{p_{Bj}}^{n_{Bj}} = C_{p_{A1}}^{n_{A1}} \oplus \cdots \oplus C_{p_{AK_A}}^{n_{AK_A}} \oplus C_{p_{B1}}^{n_{B1}} \oplus \cdots \oplus C_{p_{BK_B}}^{n_{BK_B}}$$

$\dot{A} \odot \dot{B}$

$$\bigoplus_{i=1}^{K_A} C_{p_{Ai}}^{n_{Ai}} \odot \bigoplus_{j=1}^{K_B} C_{p_{Bj}}^{n_{Bj}} = \bigoplus_{i=1}^{K_A} \bigoplus_{j=1}^{K_B} C_{p_{Ai}}^{n_{Ai}} \odot C_{p_{Bj}}^{n_{Bj}} = \bigoplus_{i=1}^{K_A} \bigoplus_{j=1}^{K_B} C_{\mathrm{lcm}(p_{Ai}, p_{Bj})}^{n_{Ai} \cdot n_{Bj} \cdot \gcd(p_{Ai}, p_{Bj})}$$

FROM THE ABSTRACTION TO A BASIC CASE…

$$\left(\bigoplus_{i=1}^{K_1} C_{p_{1i}}^{n_{1i}} \odot \dot{X}_1\right) \oplus \left(\bigoplus_{i=1}^{K_2} C_{p_{2i}}^{n_{2i}} \odot \dot{X}_2\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{K_s} C_{p_{si}}^{n_{si}} \odot \dot{X}_s\right) = \bigoplus_{j=1}^{m} C_{p_j}^{n_j}$$

Contraction steps

Intersections and Cartesian products

$$C_p^1 \odot \dot{X} = C_q^n$$

ENUMERATION PROBLEM

Solving Equations on Discrete Dynamical Systems - Dennunzio, Formenti, Margara, Montmirail, Riva. (CIBB 2019)

THE MDD-BASED PIPELINE

$$\left(\bigoplus_{i=1}^{K_1} C_{p_{1i}}^{n_{1i}} \odot \dot{x}_1^{w_1}\right) \oplus \left(\bigoplus_{i=1}^{K_2} C_{p_{2i}}^{n_{2i}} \odot \dot{x}_2^{w_2}\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{K_s} C_{p_{si}}^{n_{si}} \odot \dot{x}_s^{w_s}\right) = \bigoplus_{j=1}^{m} C_{p_j}^{n_j}$$

Necessary Equations

Identification and resolution of the basic equations.

Contractions Steps

Explorations of the feasible solutions space.

Identification of the solutions

Intersections and Unions to study the solutions of each contraction step.

W-th Roots

Algorithmic technique to compute the roots over DDSs.

THE BASIC EQUATION $C_p^1 \odot \dot{X} = C_q^n$

$p, n, q \in \mathbb{N} \setminus \{0\}$ → **EnumSOBFID** *Problem* → $\bigoplus$Solutions

$p, n, q \in \mathbb{N} \setminus \{0\}$ → $\$$ *coins system,* n *total* → ***Change-Making Problem*** → *CMP Solutions* → $\bigoplus$*Solutions*

According to the product rule: $C_p^1 \odot C_z^y = C_{lcm(p,z)}^{y \cdot \gcd(p,z)}$

A divisor r of q is in the coins system iff:

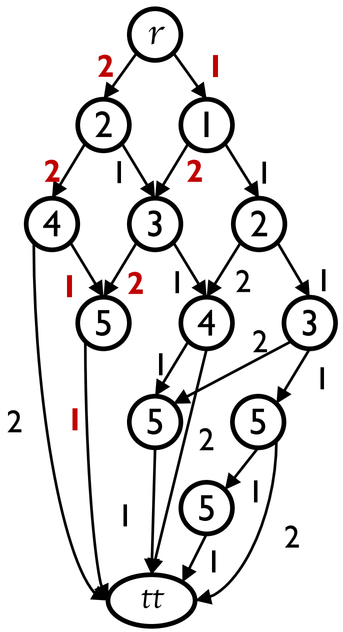
$$r \leq n \quad \text{and} \quad gcd\left(p, \frac{q}{p} \cdot r\right) = r \quad \text{and} \quad lcm\left(p, \frac{q}{p} \cdot r\right) = q$$

SB-MDDS (SYMMETRY BREAKING)

$C_2^1 \odot \dot{X} = C_6^6$ → $\$ = \{2,1\}, n = 6$ → SB−MDD $M_{2,6,6}$ → $\bigoplus Solutions$

! $r \rightarrow \frac{q}{p} \cdot r$

Reduced MDD



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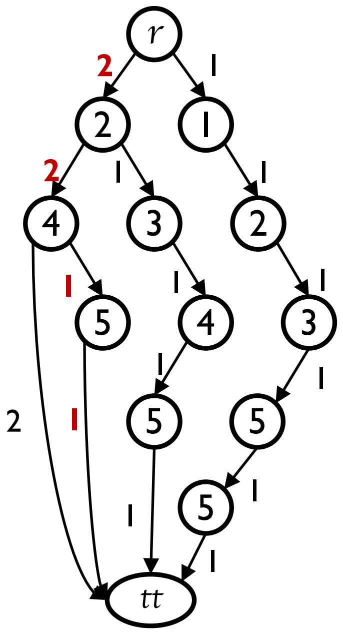

Reduced SB-MDD

$C_6^2 \oplus C_3^2$

THE MDD-BASED PIPELINE

$$\left(\bigoplus_{i=1}^{K_1} C_{p_{1i}}^{n_{1i}} \odot \dot{x}_1^{w_1}\right) \oplus \left(\bigoplus_{i=1}^{K_2} C_{p_{2i}}^{n_{2i}} \odot \dot{x}_2^{w_2}\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{K_s} C_{p_{si}}^{n_{si}} \odot \dot{x}_s^{w_s}\right) = \bigoplus_{j=1}^{m} C_{p_j}^{n_j}$$

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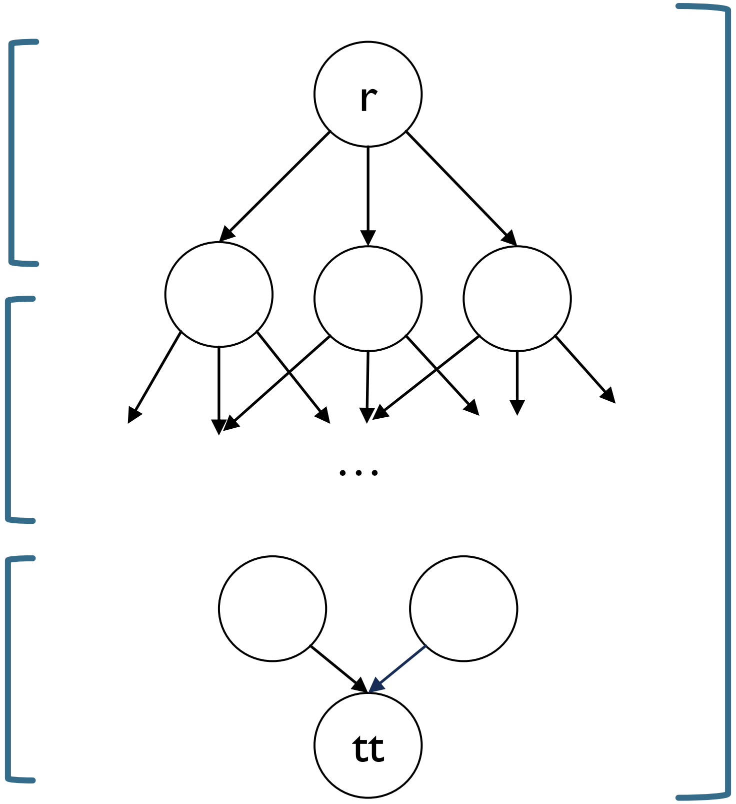
MDDS & CONTRACTIONS STEPS

$$CS = \times_{j=1}^{m} CS_j$$

For each node of a level z, the value k is a possible label of an outgoing arc if

$$M_{p_z, p_j, \frac{k}{n_z}}$$

has solution.



For each monomial there is one level into the structure + one level for the final node.

The structure represents the generation of the $C_{p_j}^{n_j}$ cycles.

EXAMPLE

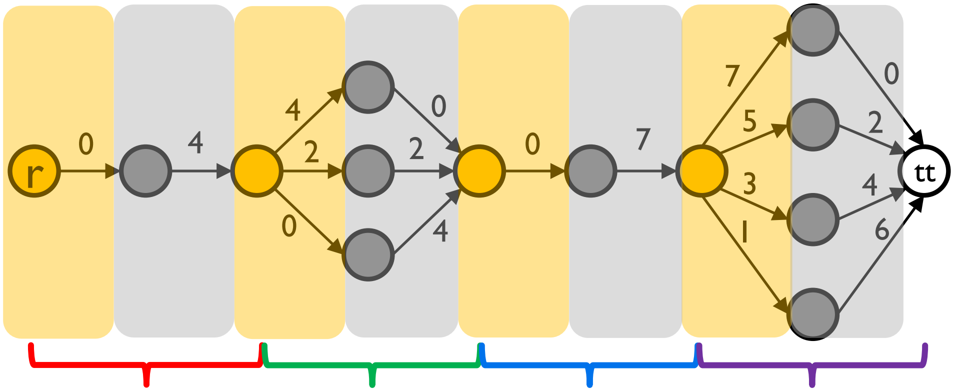
$$C_4^1 \odot \dot{X} \oplus C_2^1 \odot \dot{X} = C_2^4 \oplus C_4^4 \oplus C_6^7 \oplus C_{12}^7$$

- 33 Basic equations
- 27 Necessary equations

$C_2^1 \odot \dot{X} = C_4^1$ $\quad$ $C_2^1 \odot \dot{X} = C_{12}^1$ $\quad$ $C_2^1 \odot \dot{X} = C_{12}^5$

$C_2^1 \odot \dot{X} = C_4^3$ $\quad$ $C_2^1 \odot \dot{X} = C_{12}^3$ $\quad$ $C_2^1 \odot \dot{X} = C_{12}^7$

- Contractions steps CS



THE MDD-BASED PIPELINE

$$\left(\bigoplus_{i=1}^{K_1} C_{p_{1i}}^{n_{1i}} \odot \dot{x}_1^{w_1}\right) \oplus \left(\bigoplus_{i=1}^{K_2} C_{p_{2i}}^{n_{2i}} \odot \dot{x}_2^{w_2}\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{K_s} C_{p_{si}}^{n_{si}} \odot \dot{x}_s^{w_s}\right) = \bigoplus_{j=1}^{m} C_{p_j}^{n_j}$$

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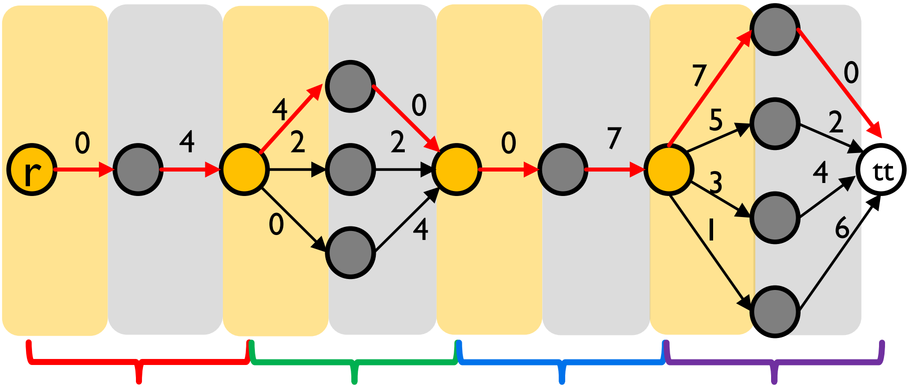
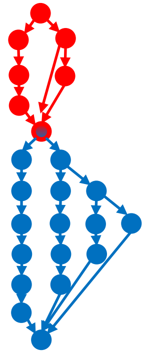
SOLVE A CONTRACTIONS STEP

Each path into CS is a feasible contractions step → Set of SB-MDDs → Same monomials → Cartesian Product → Same variable → Intersection → √

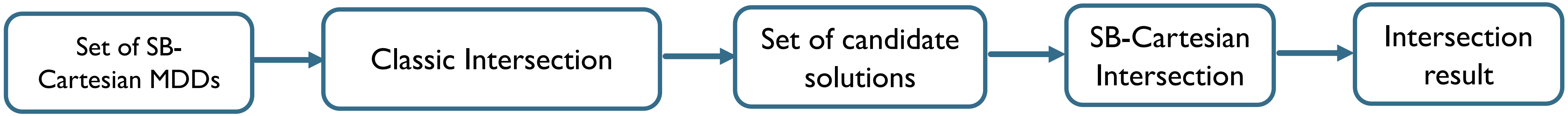


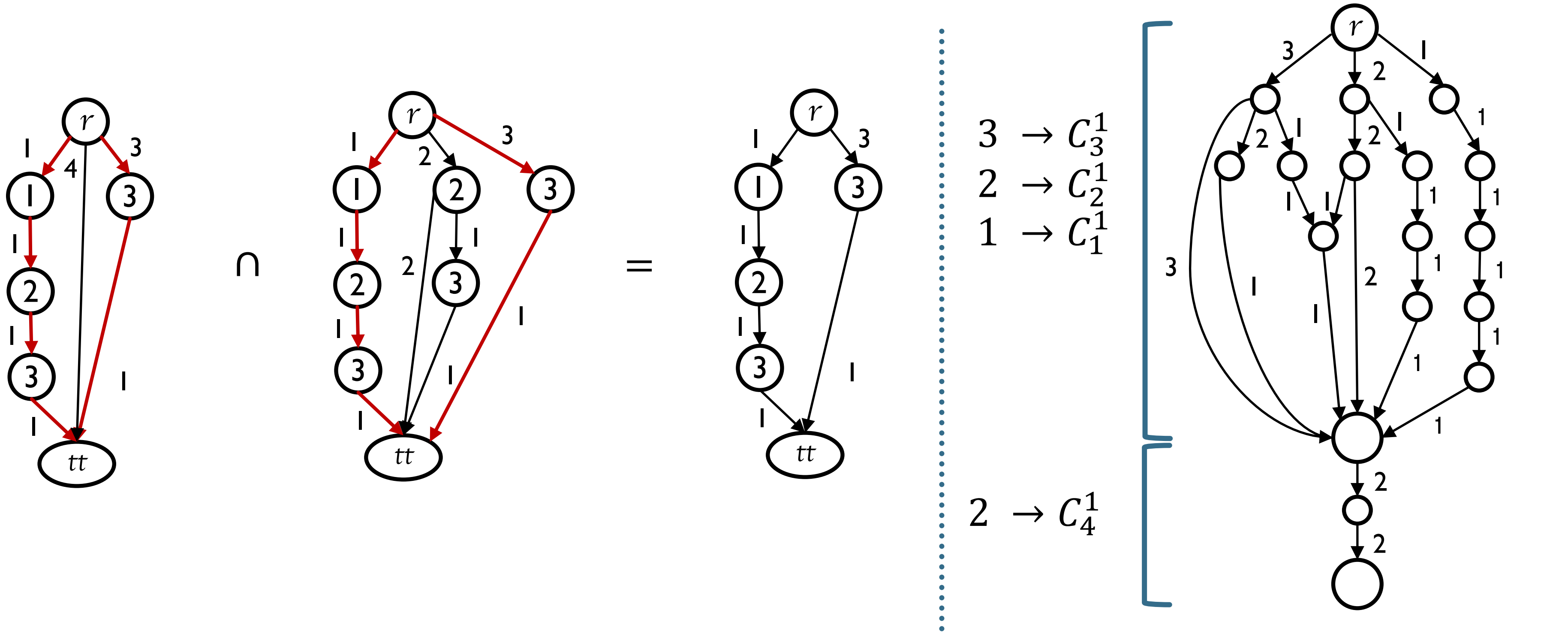
$$C_2^1 \odot \dot{X} = C_2^4 \oplus C_6^7$$



$$\begin{cases} C_2^1 \odot \dot{X} = C_2^4 \oplus C_6^7 \\ C_4^1 \odot \dot{X} = C_4^4 \oplus C_{12}^7 \end{cases}$$

INTERSECTION OF SB-MDDS





Example: [2,2,2,1,1,1,1]

!! The coins c_1, c_2 are considered equals only if

$c_1 \rightarrow C_r^1, c_2 \rightarrow C_s^1$ *and* $r = s$

THE MDD-BASED PIPELINE

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W-ROOTS OVER CYCLES

$$\dot{x}^w = C_{p_1}^{m_1} \oplus C_{p_2}^{m_2} \oplus \cdots \oplus C_{p_h}^{m_h} \quad \xrightarrow{\sqrt[w]{\ }} \quad \dot{x} = C_{q_1}^{s_1} \oplus C_{q_2}^{s_2} \oplus \cdots \oplus C_{q_l}^{s_l}$$

$$(0 < p_1 < p_2 < \ \dots < p_h)$$

Consider that some components $C_{q_1}^{s_1} \oplus \cdots \oplus C_{q_i}^{s_i}$ of $\dot{x}$ have been already computed (with $2 < i < l$).

$(C_{q_1}^{s_1} \oplus \cdots \oplus C_{q_i}^{s_i})^w = C_{p'_1}^{m'_1} \oplus C_{p'_2}^{m'_2} \oplus \cdots \oplus C_{p'_t}^{m'_t}$ with $i \le t \le h$. It holds that

q_{i+1} is $\min\{p_j \in \{p_1, \dots, p_h\} | p_j > q_i \wedge ((p_j = p'_z \wedge m'_z < m_j)$ with $z \in \{1, \dots, t\} \vee p_j \notin \{p'_1, \dots, p'_t\})\}$

s_{i+1} integer solution of

$$\sum_{\substack{k_1+k_2+\cdots+k_{i+1}=w \\ 0\le k_1,k_2,\dots,k_{i+1}\le w \\ l(q_1,\dots,q_{i+1},k_1,\dots,k_{i+1})=q_{i+1}}} \binom{w}{k_1, k_2, \dots, k_{i+1}} \prod_{\substack{t=1 \\ k_t\neq 0}}^{i+1} q_t^{k_t-1} s_t^{k_t} \prod_{\substack{t=2 \\ k_t\neq 0}}^{i+1} \gcd(l(q_1, \dots, q_{t-1}, k_1, \dots, k_{t-1}), q_t) = m_j$$

THE MDD-BASED PIPELINE

$$\left(\bigoplus_{i=1}^{K_1} C_{p_{1i}}^{n_{1i}} \odot \dot{x}_1^{w_1}\right) \oplus \left(\bigoplus_{i=1}^{K_2} C_{p_{2i}}^{n_{2i}} \odot \dot{x}_2^{w_2}\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{K_s} C_{p_{si}}^{n_{si}} \odot \dot{x}_s^{w_s}\right) = \bigoplus_{j=1}^{m} C_{p_j}^{n_j}$$

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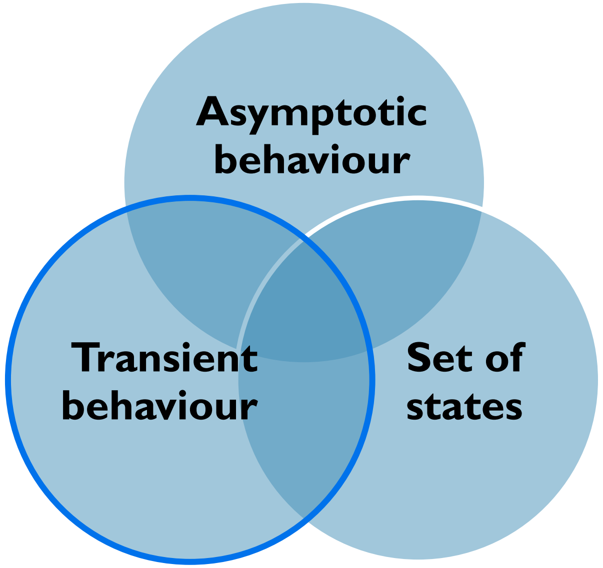
ABSTRACTION OVER THE TRANSIENT BEHAVIOUR



- Considering the abstraction over cycles, we showed that to solve the abstraction it is necessary to solve **two basic cases**:

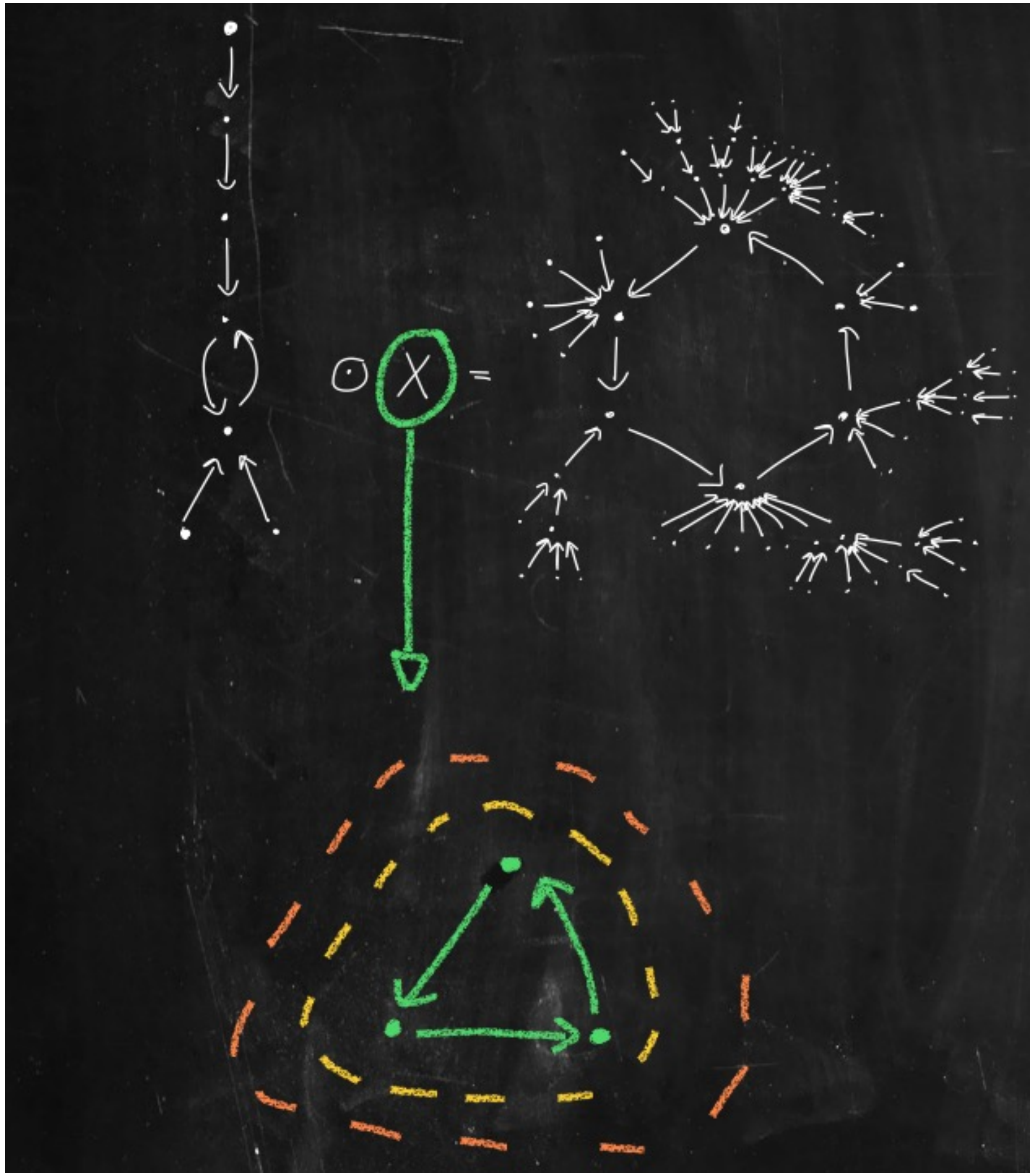
$$\alpha \cdot X = c$$

$$X^w = c$$

- **Also if we consider the transient parts** involved in the initial equation we are interested in these simple cases.

$$\alpha \cdot X = c$$

with α, c that are DDS and X with a possible cyclic behaviour.



OUR IDEA

Joint work with F. Doré and E. Formenti

CONCLUSIONS

TWO DIFFERENT ABSTRACTIONS SOLVED + ONE WORKING IN PROGRESS

A NEW **MDD APPROACH** COMPACT AND FASTER

A COMPLETE **PIPELINE** TO SOLVE GENERIC EQUATIONS OVER CYCLES

A NEW ALGORITHM TO COMPUTE **W-ROOTS** OVER THE LONG-TERM BEHAVIOUR

FUTURE DEVELOPMENTS

- Parallel solutions computation
- Improve the SB-Cartesian intersection
- Test the approach over real biological networks
- The abstraction over the transient parts of DDS
- Interaction with different abstractions

THANKS FOR YOUR ATTENTION

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