Graph Subshifts

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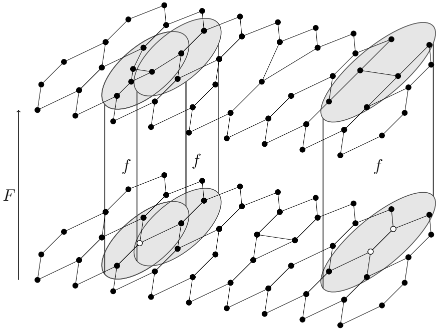
Causal graph dynamics

Causal graph dynamics (CGD) are an extension of cellular automata:

- The underlying grid becomes an arbitrary graph of bounded degree.
- The graph itself can evolve over time.



However the rule remains *local* and *shift-invariant*.

Group subshifts

Let Σ be a finite alphabet, and G be a finitely generated group.
A **configuration** is a coloring $x : G \to \Sigma$.

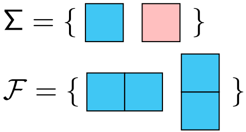
Definition

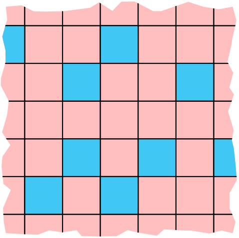
A **subshift** X is a subset of Σ^G such that there exist a set $\mathcal{F}$ of forbidden **patterns** such that :

$$X = \{x \in \Sigma^G | x \textit{ contains no patterns of } \mathcal{F}\}$$

X is said to be of **finite type** if it can be defined by a finite $\mathcal{F}$.

$G = \mathbb{Z}^2$



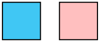


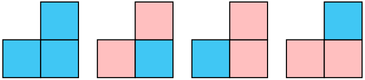
Proposition (Folklore)

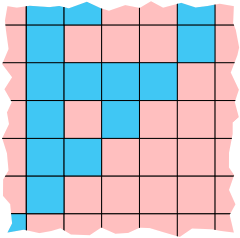
Any cellular automaton of dimension d can be seen as a subshift of finite type in dimension $d+1$.

For example, the XOR of neighborhood two can be implemented as the subshift :

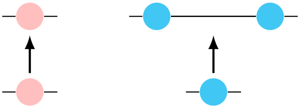
$G = \mathbb{Z}^2$

$\Sigma = \{$  $\}$

$\mathcal{F} = \{$  $\}$



Doesn't works for graph dynamics



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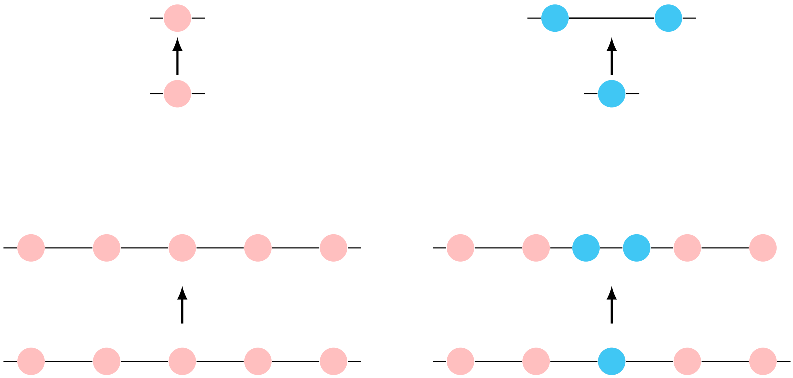

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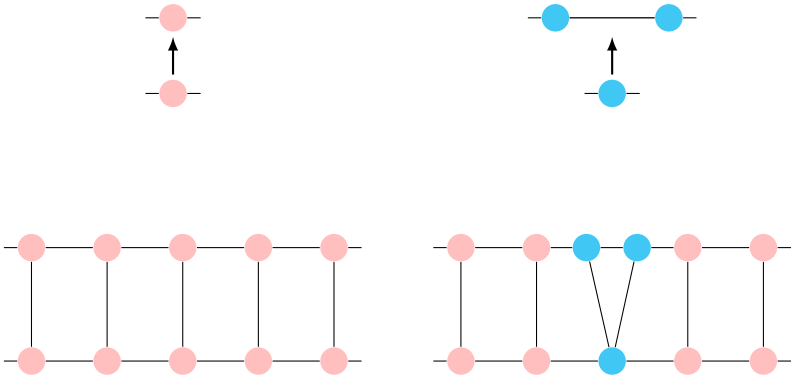
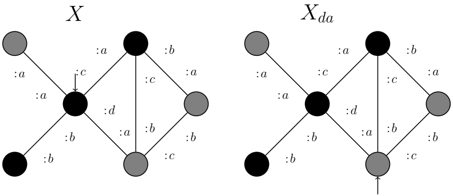


Pointed graph modulo

Definition (Configuration/Graph)

- *A set V of vertices, with a special origin vertex p the pointer*
- *A set π of ports*
- *A set E of edges, where an edge is an unordered pair $\{u : a, v : b\}$ with $u, v \in V$ and $a, b \in \pi$. A port is used atmost once.*
- *A coloring function σ from V to Σ.*

The graphs are considered modulo isomorphisms. $\mathcal{X}$ is the set of graphs.



Formalism

Definition (Cuts)

Let $\mathcal{L} \subseteq \pi^*$ be a prefix-stable language. The cut $X_{|\mathcal{L}}$ is the subgraph induced by the vertices reachable through $\mathcal{L}$.

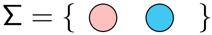
Definition (Graph Subshift)

Let $\mathcal{F}$ be a set of tuples (F, L), with F a finite graph and L a prefix-stable language. The subshift forbidding $\mathcal{F}$ is

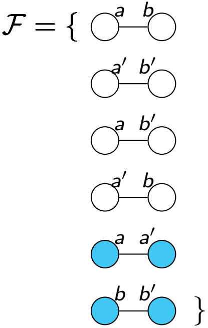
$$\mathcal{Z} = \left\{ X \in \mathcal{G}_{\Sigma,\pi} \middle| \forall v \in X, \forall (F, L) \in \mathcal{F}, (X_v)_{|L} \neq F \right\}.$$

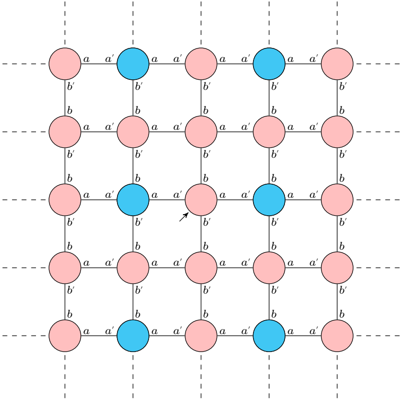
It is said to be of finite type if $\mathcal{F}$ is finite.

Example : Generalized Hardsquare model



$\pi = \{a, a', b, b'\}$





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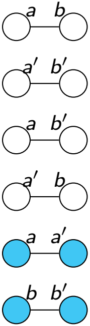
$\mathcal{F} = \{$ 

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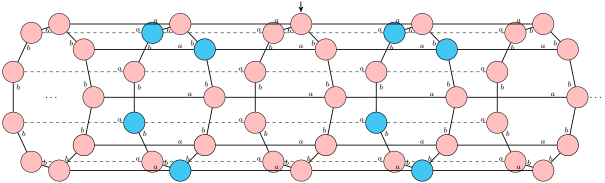

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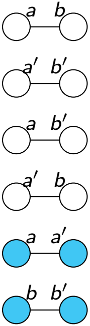
$\mathcal{F} = \{$ 

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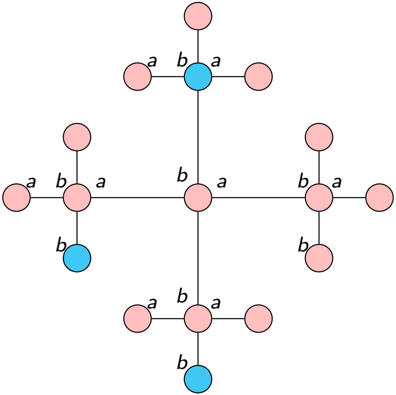

Example : Locally Grid-like

$\Sigma = \{$  $\}$

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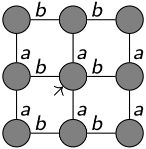
$\mathcal{F} = \mathcal{X}_{|L} \setminus \{$ 

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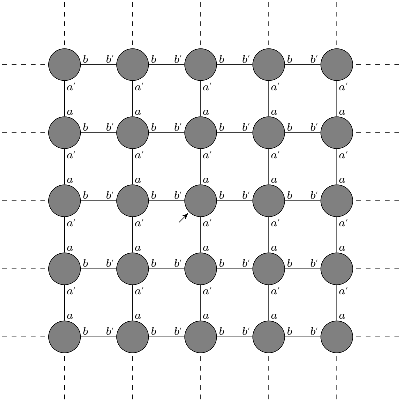

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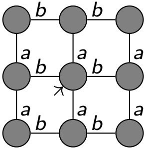
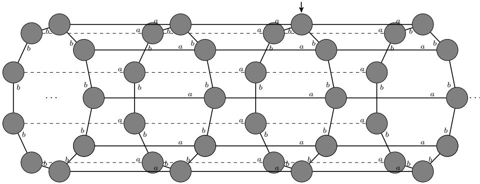
$\mathcal{F} = \mathcal{X}_{|L} \setminus \{$  $\}$



Quotient and Periodicity

Definition (Quotient)

Let X be a graph. A X' is a quotient graph *of X if there is surjective homorphism from X to X' and write $X \succeq X'$.*

Definition (Periodicity)

A period *of a graph X is a word $w \in \Pi^*$ such that $X = X_w$. The* stabilizer *of X is its set of periods.*

Definition (Strong periodicity)

A graph X is said to be strongly periodic *if its stabilizer $\textsc{stab}(X)$ is dense in X.*

Quotient and Periodicity

Proposition

The stabilizer of a configuration is a group under the concatenation operation.

Theorem

A graph is strongly periodic if and only if it admits a finite quotient.

Theorem

Let $\mathcal{Y}$ a SFT. $\mathcal{Y}$ admits a strongly periodic configuration X such that $\textsc{stab}(X)$ is residually finite if and only if it admits a finite graph.

Indecidability

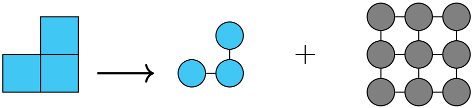
Theorem

The finite configuration problem is undecidable.

Theorem

The support graph unicity problem is undecidable.

Proof idea : Take a $\mathbb{Z}^2$ subshift and make it a locally $\mathbb{Z}^2$-like graph subshift.



Conclusion

To resume: subshifts on things that are not groups.

Perspectives: everything that has already been done in group subshifts.

Thank you for listening !