Why are reversible septenary NCCAs so simple? A preliminary inquiry

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July 13th 2021

AUTOMATA 2021

27th International Workshop on Cellular Automata and Discrete Complex Systems

Definitions

Cellular automaton:

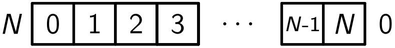
- ◇ a space of cells,
- ◇ a neighborhood,
- ◇ a state set,
- ◇ a local rule,
- ◇ a global rule.

Space of cells and the neighborhood

The grids $\mathcal{C}_N$ equal to

$$\mathcal{C}_N = \{0, 1, \ldots, N-1\}$$

the boundaries are periodically wrapped around.



We consider classical symmetrical neighborhood with radius 1:

The state set and a configuration

The set of states $Q = [0..k) = \{0, 1, 2, \ldots, k-1\}$.
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An example configuration

1	3	4	5	3	6	0	1	2	3	1	2	4	0	1	4	3	2	4	2	5

That is a function x : $\mathcal{C}_{21} \to Q$.
X_N is the set of all configurations on $\mathcal{C}_N$ and

$$X^* = \bigcup_{N=1}^{\infty} X_N$$ — the set of all finite configurations

A local and global rule

A local rule is a function $f : [0..k)^3 \to [0..k)$, often described as a string of numbers (LUT). For example f(000) f(001) ... f(666) for a septenary CA. A given local rule f induces a *global rule* $F : X^* \to X^*$ in the usual way

$$F(\mathrm{x})_n = f(x_{n-1}, x_n, x_{n+1})$$

Introducing time steps we get a space-time diagram:

1	3	4	5	3	6	0	1	2	3	1	2	4	0	1	4	3	2	4	2	5
1	1	6	5	5	0	4	3	0	3	3	4	0	0	3	2	5	4	0	2	5
1	3	4	5	1	4	4	1	2	3	1	6	0	0	5	0	3	6	0	2	5
1	1	6	5	3	2	4	3	0	3	3	0	4	0	1	4	5	0	4	2	5
1	3	4	5	5	4	0	1	2	3	1	2	4	0	3	2	1	4	4	2	5
1	1	6	5	3	6	0	3	0	3	3	4	0	0	5	0	3	2	4	2	5
1	3	4	5	5	0	4	1	2	3	1	6	0	0	1	4	5	4	0	2	5
1	1	6	5	1	4	4	3	0	3	3	0	4	0	3	2	3	6	0	2	5
1	3	4	5	3	2	4	1	2	3	1	2	4	0	5	0	5	0	4	2	5
1	1	6	5	5	4	0	3	0	3	3	4	0	0	1	4	1	4	4	2	5
1	3	4	5	3	6	0	1	2	3	1	6	0	0	3	2	3	2	4	2	5
1	1	6	5	5	0	4	3	0	3	3	0	4	0	5	0	5	4	0	2	5
1	3	4	5	1	4	4	1	2	3	1	2	4	0	1	4	3	6	0	2	5
1	1	6	5	3	2	4	3	0	3	3	4	0	0	3	2	5	0	4	2	5
1	3	4	5	5	4	0	1	2	3	1	6	0	0	5	0	1	4	4	2	5

Number conservation and reversibility

A global rule F is

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k	all k-ary CAs	k-ary NCCAs	reversible k-ary CAs	reversible k-ary NCCAs
2	$2^{2^3} = 256$	5	6	3
3	$3^{3^3} \approx 7.6 \cdot 10^{12}$	144	1800	3
4	$4^{4^3} \approx 3.4 \cdot 10^{38}$	89 588	?	21
5	$5^{5^3} \approx 2.4 \cdot 10^{87}$	1 876 088 314	?	21
6	$6^{6^3} \approx 1.2 \cdot 10^{168}$	?	?	471
7	$7^{7^3} \approx 7.4 \cdot 10^{289}$	?	?	1669

Table 1: Numbers of specific types of one-dimensional k-ary CAs with radius 1.

Quinary (5 states)

21 rules

F	Description of F	F	Description of F	F	Description of F
1	the right-shift rule	8	$(31)\leftrightarrow(40), (32)\leftrightarrow(41)$	15	$(04)\leftrightarrow(13), (14)\leftrightarrow(23)$
2	$(12)\leftrightarrow(30)$	9	the identity rule	16	$(04)\leftrightarrow(31)$
3	$(12)\leftrightarrow(30), (14)\leftrightarrow(32)$	10	$(32)\leftrightarrow(41)$	17	$(03)\leftrightarrow(12)$
4	$(21)\leftrightarrow(30), (31)\leftrightarrow(40)$	11	$(23)\leftrightarrow(41)$	18	$(03)\leftrightarrow(12), (04)\leftrightarrow(13)$
5	$(21)\leftrightarrow(30)$	12	$(14)\leftrightarrow(23)$	19	$(03)\leftrightarrow(21)$
6	$(13)\leftrightarrow(40)$	13	$(14)\leftrightarrow(32)$	20	$(03)\leftrightarrow(21), (23)\leftrightarrow(41)$
7	$(31)\leftrightarrow(40)$	14	$(04)\leftrightarrow(13)$	21	the left-shift rule

Table 2: The list of all reversible quinary NCCAs with radius 1. The global rule of a given CA is described in terms of swaps, where a swap $(ab) \leftrightarrow (cd)$ means that every pattern '*ab*' in the subsequent time step is replaced by the pattern '*cd*' and vice versa.

Remark. For each F except shifts we have $F^2 = id$.

Senary (6 states)

471 rules, described as an Open Access Dataset:

DOI: 10.34808/b8pn-1523

MOST WIEDZY what are you looking for? open

Main Page > Open Research Data > T > The complete lists of 1D reversible number-conserving cellular automata with radius one of up to 7 s...

The complete lists of 1D reversible number-conserving cellular automata with radius one of up to 7 states

Description

This dataset contains complete lists of all one-dimensional reversible number-conserving k-ary cellular automata with radius one of up to 7 states, i.e. with state sets {0,1}, {0,1,2}, {0,1,2,3}, {0,1,2,3,4}, {0,1,2,3,4,5} and {0,1,2,3,4,5,6}.

The detailed definitions and the method of enumerating are given in the paper:

Barbara Wolnik, Maciej Dziemiańczuk, Adam Dzedzej, and Bernard De Baets "Reversibility of number-conserving 1D cellular automata: unlocking insights into the dynamics for larger state sets".

For a given number of states k, we use m=k-1 as the largest state. The list of all one-dimensional reversible number-conserving k-ary cellular automata with radius one with state set {0,1,...,m} is given in the look up table format, i.e., each rule f is given as the string

f(000) f(001) ... f(00m) f(010) ... f(01m) ... f(mm0) f(mm1) ... f(mmm)

of k^3 integers from the set {0,1, ..., m}.

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Septenary (7 states)

1669 rules

m	1	2	3	4	6	12	30	60
	1	634	72	324	540	8	60	28

Table 3: The number of reversible septenary NCCAs that have order m.

Each initial configuration, irrespective of size, cycles after 60 moves!

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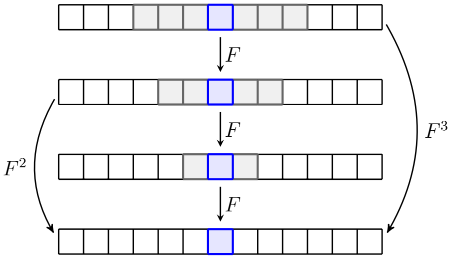


radius of F^k in general is k

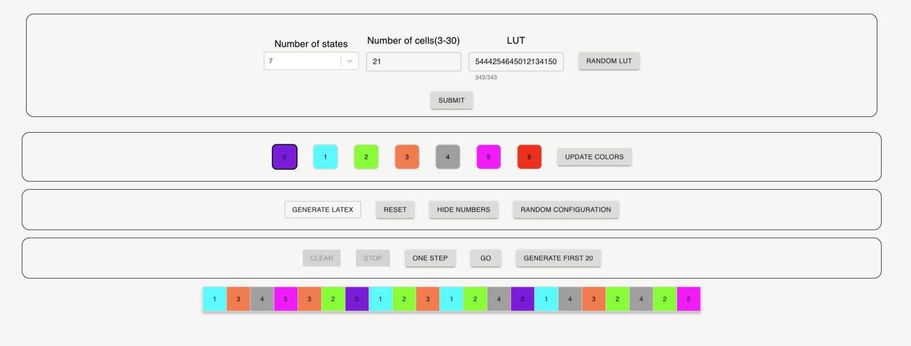
Space time diagrams comparison

JitTeam™
the human factor
of IT

Aleksander Wardyn's application showtime.

https://automata-1d.herokuapp.com/

Application



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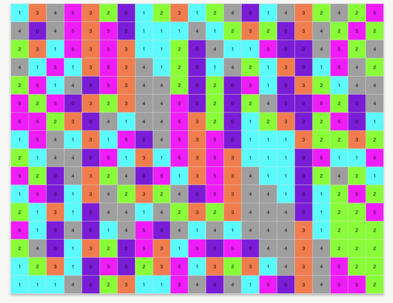
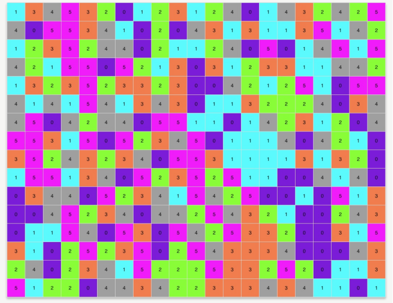
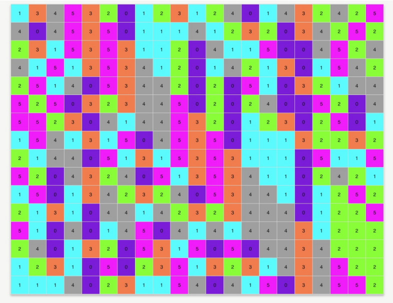
Space diagrams for senary rules

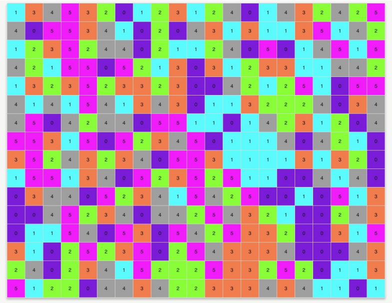




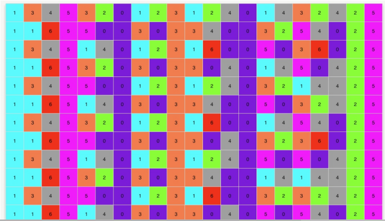
Comparison

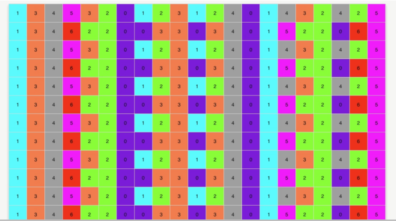
Space diagrams for senary rules





Space diagrams for septenary rules





Rule 2 detailed description

LUT of Rule 2:

0000000	1131313	2222622	1353135	4444444	1555355	2666666
0000000	1131313	0000400	1353135	2222222	1555355	0444444
0000000	1131313	2222622	1353135	0000000	1555355	2666666
2222222	1131313	0000400	1353135	6666666	1555355	0444444
0000000	1131313	2222622	1353135	4444444	1555355	2666666
4444444	1131313	2222622	1353135	6666666	1555355	2666666
4444444	1131313	2222622	1353135	4444444	1555355	2666666

Some technical lemmas specific to this rule, no general method for description.

Rule 2 detailed description

$$\mathcal{A} = \begin{bmatrix} 3 & 6 & 0 \\ 5 & 0 & 4 \\ 1 & 4 & 4 \\ 3 & 2 & 4 \\ 5 & 4 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1 & 6 & 0 \\ 3 & 0 & 4 \\ 1 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 3 & 2 \\ 5 & 0 \\ 1 & 4 \end{bmatrix}, \tag{1}$$

$$\mathcal{D} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} 3 & 6 \\ 5 & 4 \end{bmatrix}.$$

$$t=0 \quad \boxed{1}\,\underbrace{\boxed{3}\,\boxed{4}}_{\mathcal{E}_2}\,\boxed{5}\,\underbrace{\boxed{3}\,\boxed{6}\,\boxed{0}}_{\mathcal{A}_1}\,\underbrace{\boxed{1}\,\boxed{2}}_{\mathcal{D}_2}\,\boxed{3}\,\underbrace{\boxed{1}\,\boxed{2}\,\boxed{4}}_{\mathcal{B}_3}\,\boxed{0}\,\underbrace{\boxed{1}\,\boxed{4}}_{\mathcal{C}_3}\,\underbrace{\boxed{3}\,\boxed{2}\,\boxed{4}}_{\mathcal{A}_4}\,\boxed{2}\,\boxed{5}$$

$$\Big\downarrow \left(\texttt{Rule2}\right)^{2021}$$

$$t=2021 \quad \boxed{1}\,\underbrace{\boxed{1}\,\boxed{6}}_{\mathcal{E}_1}\,\boxed{5}\,\underbrace{\boxed{5}\,\boxed{0}\,\boxed{4}}_{\mathcal{A}_2}\,\underbrace{\boxed{3}\,\boxed{0}}_{\mathcal{D}_1}\,\boxed{3}\,\underbrace{\boxed{3}\,\boxed{4}\,\boxed{0}}_{\mathcal{B}_4}\,\boxed{0}\,\underbrace{\boxed{5}\,\boxed{0}}_{\mathcal{C}_2}\,\underbrace{\boxed{5}\,\boxed{4}\,\boxed{0}}_{\mathcal{A}_5}\,\boxed{2}\,\boxed{5}$$

Main open Question

Question

Let k be a prime number and F be the global rule of a reversible NCCA with radius 1 and state set $\{0, 1, \ldots, k-1\}$*. If F is not a shift, does it necessarily have finite order?*

The answer is true for 2,3,5 and 7 and will be difficult to check for the next prime, which is 11.
For composite numbers there exist methods (for example using a two-layer representation) that allow to construct reversible *k*-ary NCCAs with radius 1 having infinite order (other than shifts).

Final slide

Dziękujemy za uwagę!
Merci pour votre attention!
Thank you for your attention!
Vielen Dank für Ihre Aufmerksamkeit!
Grazie per l'attenzione!