Cellular automata and substitutions in the edit-distance space

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Definitions and notations

- A finite set A is called an **alphabet** and its elements are called **letters**.
- A **finite** (resp. **infinite**) word over an alphabet A is the concatenation of finite (resp. infinite) number of letters of A.
- The **length** of finite word u is denoted by $|u|$. The only word with length null is the empty word denoted by λ.
- The set of finite (resp. infinite) words denoted by A^* (resp. $A^{\mathbb{N}}$).
- The set of words of the same length $n \in \mathbb{N}$ is denoted by A^n.
- For $x \in A^{\mathbb{N}}$ and for $i,j \in \mathbb{N}$, we denote by x_i the $i+1$ element of x and $x_{[\![i,j[\![} = x_i x_{i+1} \ldots x_{j-1}$.
- The **shift-map** denoted by σ and defined over the set of infinite words by :

$$\sigma(x)_i = x_{i+1}, \quad \forall x \in A^{\mathbb{N}}.$$

Definition : [Cattaneo, Formenti, Margara, and Mazoyer. 1997]

The **Besicovitch pseudo-metric**, denoted here by $d_{\mathcal{B}}$, is defined as follows :

$$d_{\mathcal{B}}(x,y) = \limsup_{l\to\infty} \frac{d_H(x_{[\![0,l[\![}, y_{[\![0,l[\![})}{l}, \ \ \forall x,y \in A^{\mathbb{N}}.$$

With d_H represent the **Hamming distance** defined by :

$$d_H(u,v) = |\{ i \in [\![0,|u|[\![\mid u_i \neq v_i \}|, \ \forall u,v \in A^* \ \ and \ \ |u| = |v|.$$

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	......
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	......

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	……
	|	|	|	|	|	|	|	|	|	|										
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	……

$$\frac{d_H(x_{[\![0,11[\![}, y_{[\![0,11[\![})}{10} = \frac{5}{10}$$

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	......
	|	|	|	|	|	|	|	|	|	|	|									
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	......

$$\frac{d_H(x_{\llbracket 0,12 \llbracket}, y_{\llbracket 0,12 \llbracket})}{11} = \frac{6}{11}$$

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0 ......
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1 ......

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	......
	|	|	|	|	|	|	|	|	|	|	|	|	|							
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	......

$$\frac{d_H(x_{[\![0,14[\![}, y_{[\![0,14[\![})}{13} = \frac{7}{13}$$

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	……
	|	|	|	|	|	|	|	|	|	|	|	|	|	|						
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	……

$$\frac{d_H(x_{[\![0,15[\![}, y_{[\![0,15[\![})}{14} = \frac{7}{14}$$

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Example

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	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|					
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	......

$$\frac{d_H(x_{[\![0,16[\![}, y_{[\![0,16[\![})}{15} = \frac{8}{15}$$

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	......
	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|				
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	......

$$\frac{d_H(x_{[\![0,17[\![}, y_{[\![0,17[\![})}{16} = \frac{8}{16}$$

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	......
	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|			
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	......

$$\frac{d_H(x_{[\![0,18[\![}, y_{[\![0,18[\![})}{17} = \frac{9}{17}$$

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Example

x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	......
	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|	|	
y :	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	1	......

$$d_{\mathcal{B}}(x,y) = \frac{1}{2}$$

Definition

Let A be an alphabet, and, let $X = A^{\mathbb{N}}$. The relation :

$$x \sim_{\mathcal{B}} y \Longleftrightarrow d_{\mathcal{B}}(x,y) = 0,$$

is an equivalence relation. The quotient space $X_{/\sim_{\mathcal{B}}}$ is a metric space called the ***Besicovitch space*** *denoted by $X_{\mathcal{B}}$. We denote by $\tilde{x}$ the equivalence class of $x \in A^{\mathbb{N}}$ in the quotient space.*

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Topological properties : [Blanchard, Formenti, Kurka 1997]

The Besicovitch space is pathwise-connected, infinite-dimension and complete topological space, but, it is neither separable nor locally compact.

Definition

*A **cellular automaton** of radius r is a map $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$, such that there exists a map $f : A^{r+1} \to A$, for all $x \in A^{\mathbb{N}}$, $i \in \mathbb{N}$: $F(x)_i = f(x_{[\![i,i+r[\![})$. The map f is called **the local rule** of F.*

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The **XOr** CA, defined by : $f(ab) = a + b \mod 2, \ \forall a, b \in \{0, 1\}$.

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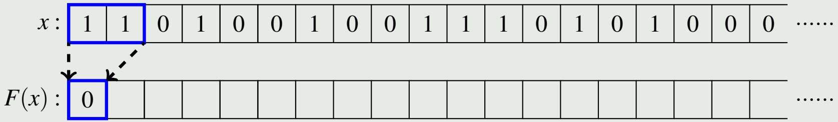

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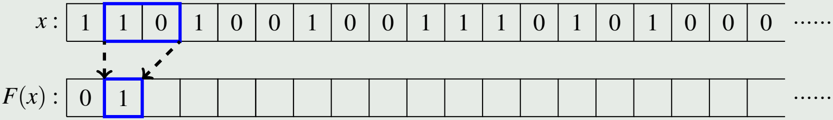

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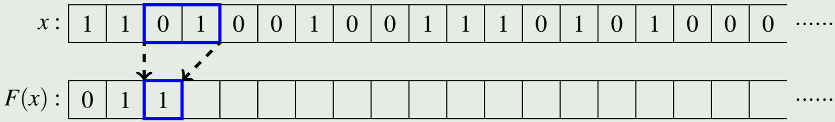

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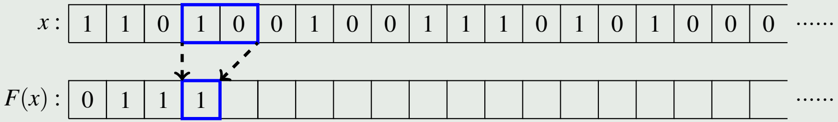

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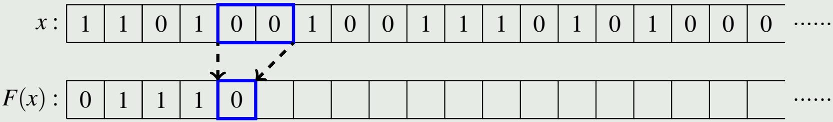

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x :	1	1	0	1	0	0	1	0	0	1	1	1	0	1	0	1	0	0	0	······
$F(x)$:	0	1	1	1	0	1	1	0	1	0	0	1	1	1	1	1	0	0		······

Definition

A function $\tilde{F}$ is a cellular automaton on the Besicovitch space if there exists a cellular automaton $G : A^{\mathbb{N}} \to A^{\mathbb{N}}$ such that $G \in \tilde{F}$ (i.e. $\forall x \in A^{\mathbb{N}}, \forall y \in \tilde{F}(\tilde{x}),\ d_{\mathcal{B}}(G(x), y) = 0$).

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Theorems : [Blanchard, Formenti, and Kurka 1997]

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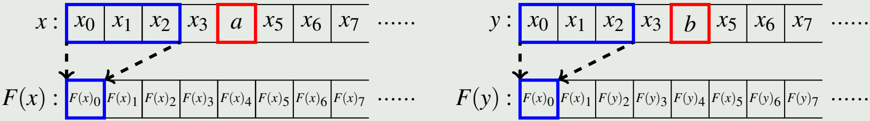
Let F be a CA with radius $r = 2$, $x, y \in A^{\mathbb{N}}$ and $a \neq b \in A$:



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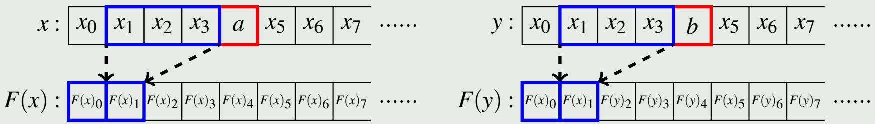

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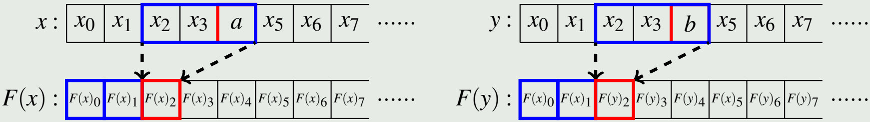

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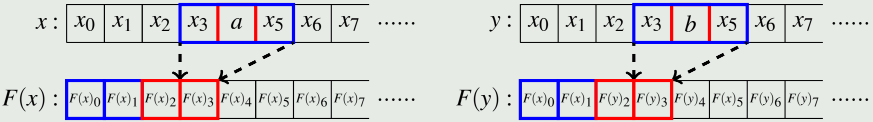

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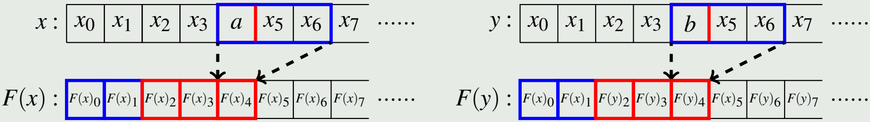

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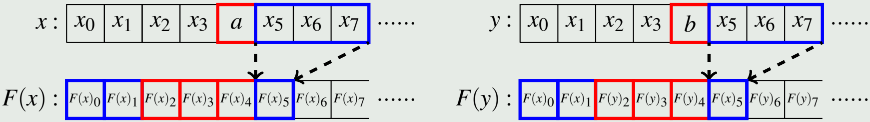

Definition

A function $\tilde{F}$ is a cellular automaton on the Besicovitch space if there exists a cellular automaton $G : A^{\mathbb{N}} \to A^{\mathbb{N}}$ such that $G \in \tilde{F}$ (i.e. $\forall x \in A^{\mathbb{N}}, \forall y \in \tilde{F}(\tilde{x}),\ d_{\mathcal{B}}(G(x), y) = 0$).

Theorems : [Blanchard, Formenti, and Kurka 1997]

Every CA induces a (well-defined) Lipshitz function over Besicovitch space.

Example

Let F be a CA with radius $r = 2$, $x, y \in A^{\mathbb{N}}$ and $a \neq b \in A$:

x :	x_0	x_1	x_2	x_3	a	x_5	x_6	x_7	......
$F(x)$:	$F(x)_0$	$F(x)_1$	$F(x)_2$	$F(x)_3$	$F(x)_4$	$F(x)_5$	$F(x)_6$	$F(x)_7$	......

y :	x_0	x_1	x_2	x_3	b	x_5	x_6	x_7	......
$F(y)$:	$F(x)_0$	$F(x)_1$	$F(y)_2$	$F(y)_3$	$F(y)_4$	$F(x)_5$	$F(y)_6$	$F(y)_7$	......

$$d_{\mathcal{B}}(F(x), F(y)) \leq (r+1) \times d_{\mathcal{B}}(x, y).$$

Definitions

Let A be an alphabet.

1. *A **substitution** τ is a non-erasing morphism over A^*, i.e. τ replaces the letters of an alphabet A with non-empty finite words.*
2. *The application associated to a substitution τ denoted by $\overline{\tau}$ and defined on $A^{\mathbb{N}}$ by :*

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Examples

1. The **Fibonnacci** substitution defined on $A = \{0,1\}$ by :

$$\tau : \begin{array}{ll} 0 & \longrightarrow 01 \\ 1 & \longrightarrow 0 \end{array}$$

2. The **Thue Morse** substitution defined on $A = \{0,1\}$ by :

$$\tau : \begin{array}{ll} 0 & \longrightarrow 01 \\ 1 & \longrightarrow 10 \end{array}$$

Definition

A substitution τ *is called* ***uniform*** *if for all* $a, b \in A$*,* $|\tau(a)| = |\tau(b)|$*. The length of a uniform substitution is* $|\tau(a)|$ *with* $a \in A$*.*

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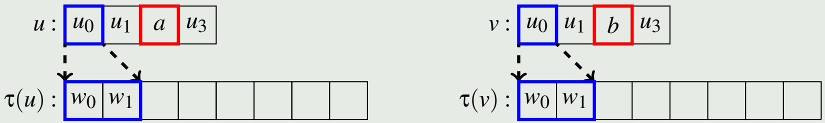

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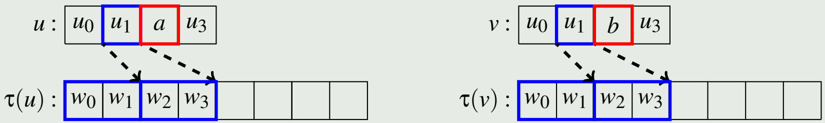

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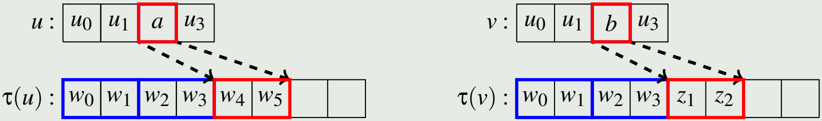

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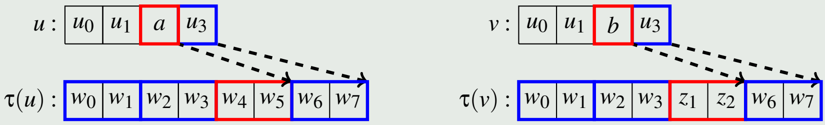

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Example

Let τ be a uniform substitution with length $L = 2$ and $a \neq b \in A$:

$u : \boxed{u_0 \mid u_1 \mid a \mid u_3}$ $\qquad$ $v : \boxed{u_0 \mid u_1 \mid b \mid u_3}$

$\tau(u) : \boxed{w_0 \mid w_1 \mid w_2 \mid w_3 \mid w_4 \mid w_5 \mid w_6 \mid w_7}$ $\qquad$ $\tau(v) : \boxed{w_0 \mid w_1 \mid w_2 \mid w_3 \mid z_1 \mid z_2 \mid w_6 \mid w_7}$

$$d_H(\tau(u), \tau(v)) \leq L \times d_H(u, v).$$

Theorem

For every non-uniform substitution τ, $\overline{\tau}$ *is well-defined on the Besicovitch space iff it is constant.*

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Examples

1. Let τ be the substitution defined on $A = \{0, 1\}$ by :

$$\tau : \begin{array}{ll} 0 & \longrightarrow 01 \\ 1 & \longrightarrow 0101 \end{array}$$

$\overline{\tau}$ is constant (i.e $\forall x \in A^{\mathbb{N}},\ \overline{\tau}(x) = (01)^{\infty}$), so it is well defined over the Besicovitch space.

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2. Let τ be the Fibonacci substitution defined on $A = \{0, 1\}$ by :

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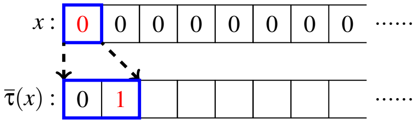
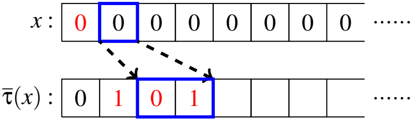
Let $x = 0^{\infty}$ and $y = 10^{\infty}$ so $d_{\mathcal{B}}(x, y) = 0$. Then $\tilde{x} = \tilde{y}$.


x : 0 0 0 0 0 0 0 0 ......
τ̄(x) : 0 1 ......
y : 1 0 0 0 0 0 0 0 ......
τ̄(y) : 0 ......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0 0 0 0 0 0 0 0 ......
τ̄(x) :
0 1 0 1 ......
y :
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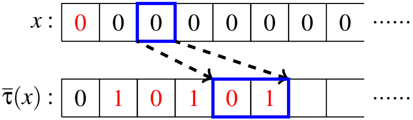
x : 0 0 0 0 0 0 0 0 ......
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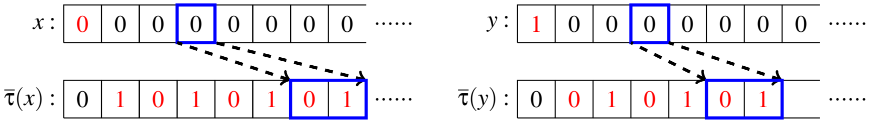
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τ̄(x) :
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y :
1 0 0 0 0 0 0 0 ......
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0 0 1 0 1 0 1 ......

x :	0	0	0	0	0	0	0	0	……
$\overline{\tau}(x)$:	0	1	0	1	0	1	0	1	……

y :	1	0	0	0	0	0	0	0	……
$\overline{\tau}(y)$:	0	0	1	0	1	0	1		……

$$\overline{\tau}(x) = (01)^{\infty} \text{ and } \overline{\tau}(y) = 0(01)^{\infty} \Longrightarrow d_{\mathcal{B}}(\overline{\tau}(x), \overline{\tau}(y)) = 1$$

$$\Longrightarrow \widetilde{\overline{\tau}(x)} \neq \widetilde{\overline{\tau}(y)}.$$

Definitions

The edit operations are defined over A^* *as follows, for* $u \in A^*$, $a \in A$ *and* $i \in [\![0, |u|[\![$ *:*

- *The **deletion** of a letter :* $D_i(u) = u_0 u_1 \dots u_{i-1}\ u_{i+1} \dots u_{|u|-1}$.
- *The **letter-switch** :* $S_i^a(u) = u_0 u_1 \dots u_{i-1}\ a\ u_{i+1} \dots u_{|u|-1}$.
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Definition :

The **edit distance** between two finite words u and v, denoted by $d_{\mathcal{L}}(u,v)$, is defined as the minimal $n \in \mathbb{N}$ such that $T_1 \circ \dots \circ T_n(u) = v$ for some edit operations $(T_k)_{1 \leq k \leq n}$.

Definitions

*The **centred pseudo-metric** associated to the edit distance is :*

$$\mathfrak{C}_{d_{\mathcal{L}}}(x,y) = \limsup_{l\to\infty} \frac{d_{\mathcal{L}}(x_{[\![0,l[\![}, y_{[\![0,l[\![})}{l}, \;\; \forall x,y \in A^{\mathbb{N}}.$$

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Example

x :	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	......
y :	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	......

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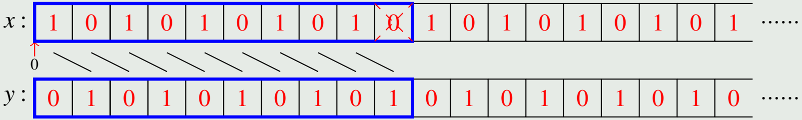

$$\frac{d_{\mathcal{L}}(x_{[\![0,11[\![}, y_{[\![0,11[\![})}{10} = \frac{2}{10} \quad and \quad \frac{d_H(x_{[\![0,11[\![}, y_{[\![0,11[\![})}{10} = \frac{10}{10}$$

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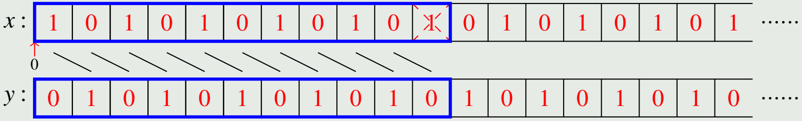

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Example

x :	1	0	1	0	1	0	1	0	1	0	1	~~0~~	1	0	1	0	1	0	1	……
y :	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	……

0

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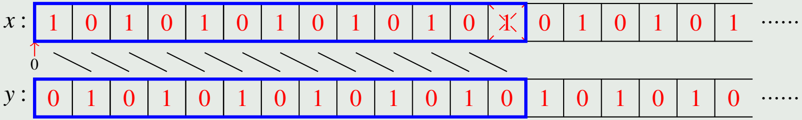

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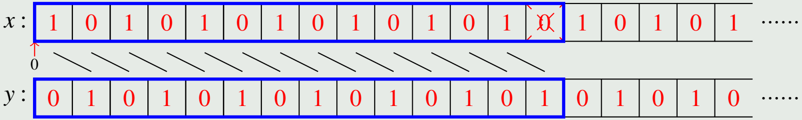

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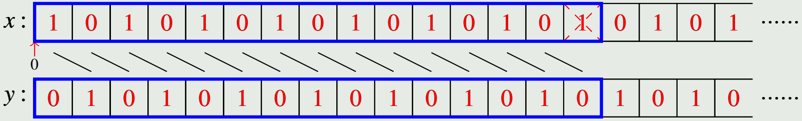

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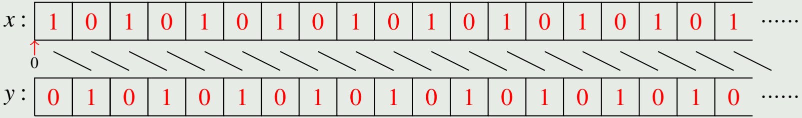

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Example

$x:$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	......
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↑ 0

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Definition

*We quotient the space of infinite words by the equivalence of zero distance to find a metric space called **centred space associated to edit distance** denoted by $X_{\mathcal{L}}$ where $X = A^{\mathbb{N}}$.*

Proposition

The topology induced by the pseudo-distance associated to the edit distance is finer then Besicovitch topology.

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Remark

Since every class is invariant by shift, dynamical systems over this space can be considered as acting on shift orbits.

Theorem

Every CA is Lipschitz with respect to $\mathfrak{C}_{d_{\mathcal{L}}}$. In particular it is well defined on the quotient space.

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Notations

Let F be CA with radius r and local rule f. We denote by f^* the function defined over A^* by :

$$f^*(u) = \begin{cases} f(u_{[\![0,r[\![})f(u_{[\![1,1+r[\![})\dots f(u_{[\![|u|-r,|u|[\![}) & \textit{if} \quad |u| \geq r. \\ \lambda & \textit{if} \quad |u| < r. \end{cases}$$

Lemma

Let F be a CA with radius r and local rule f. Then for all $u \in A^{r+1}A^*, v \in A^r A^*$, and for some edit operation T, we have :

$$d_{\mathcal{L}}(f^*(T(u)),f^*(v)) \leq r+1+d_{\mathcal{L}}(f^*(u),f^*(v)).$$

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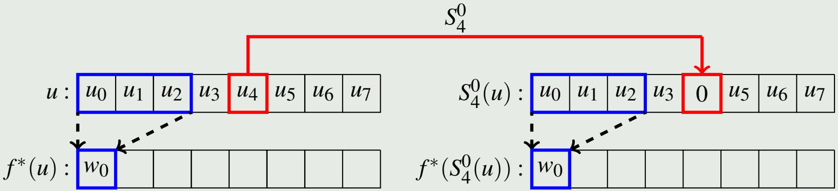

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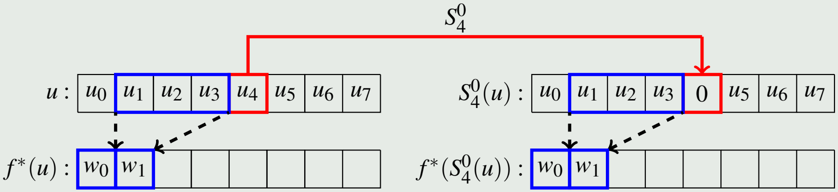

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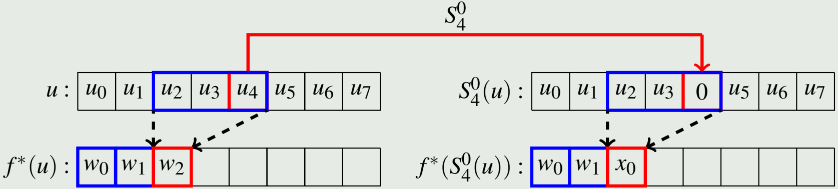

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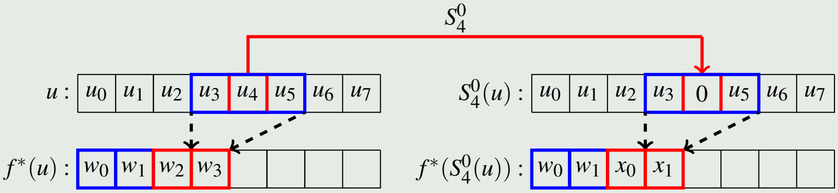

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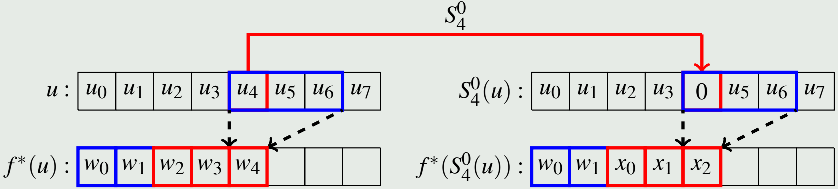

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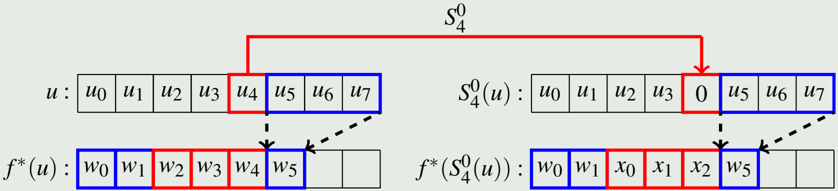

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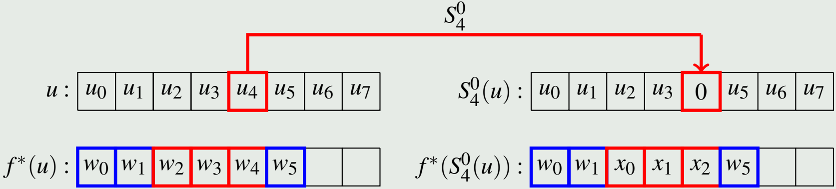

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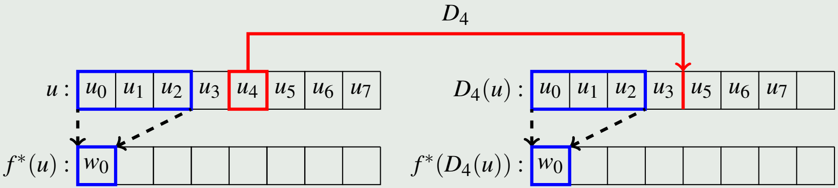

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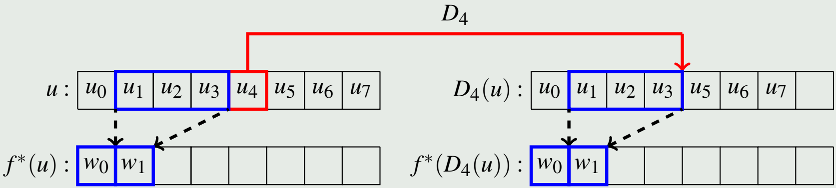

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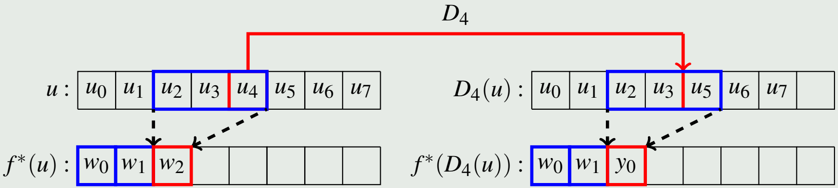

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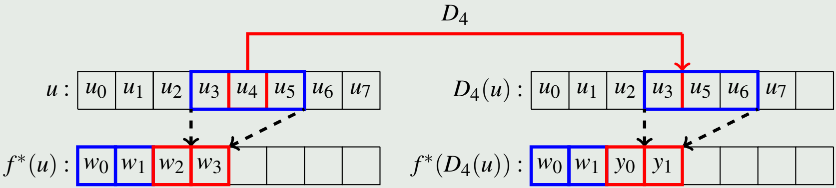

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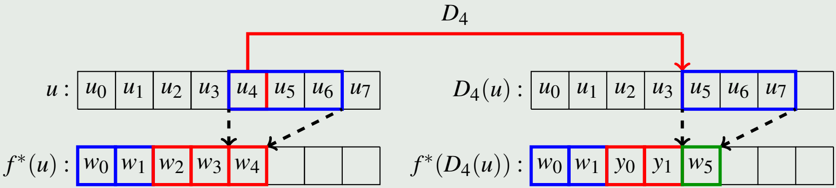

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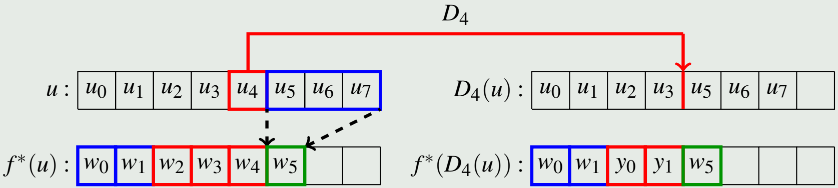

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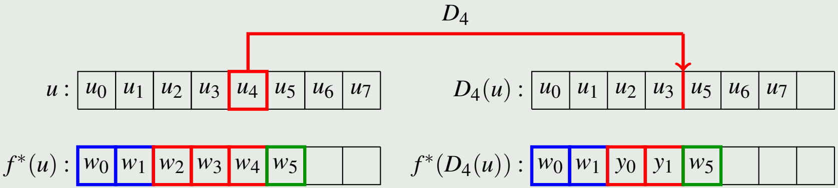

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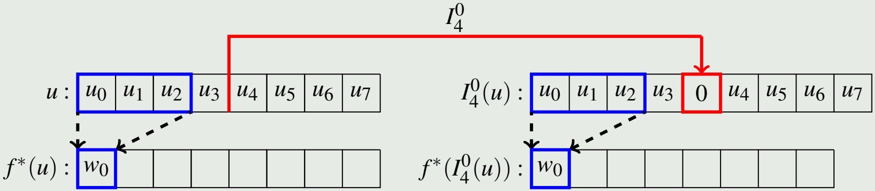

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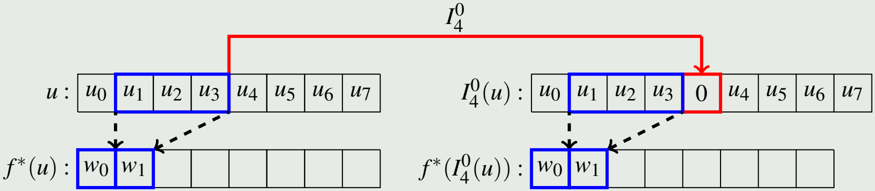

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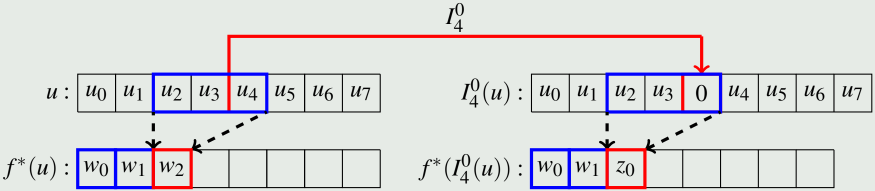

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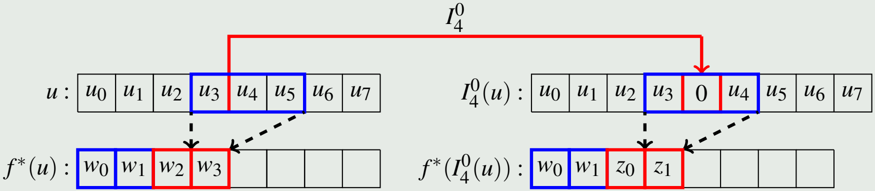

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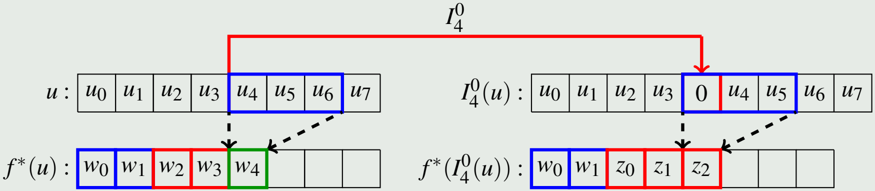

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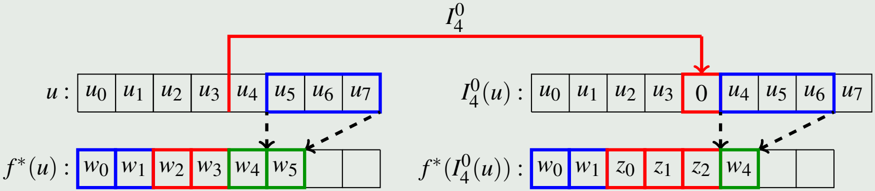

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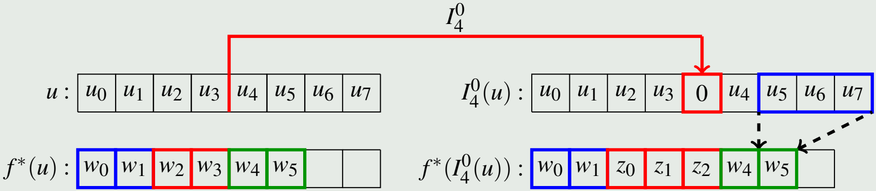

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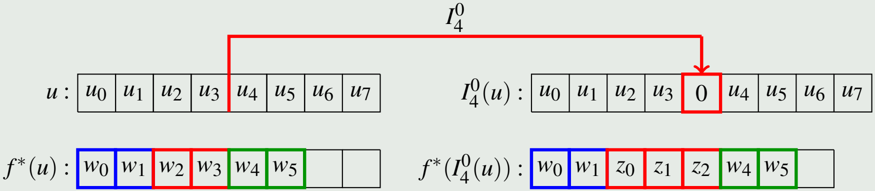

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Let F be a CA with radius r and local rule f. Then for all $u, v \in A^rA^*$ such that $d_{\mathcal{L}}(u, v) \leq |u| - r$, we have :

$$d_{\mathcal{L}}(f^*(u), f^*(v)) \leq (r + 1)d_{\mathcal{L}}(u, v).$$

Theorem

For every substitution τ, the map $\overline{\tau}$ is well-defined over $X_{\mathcal{L}}$. Furthermore it is Lipschitz.

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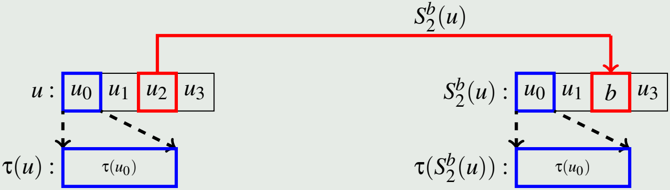

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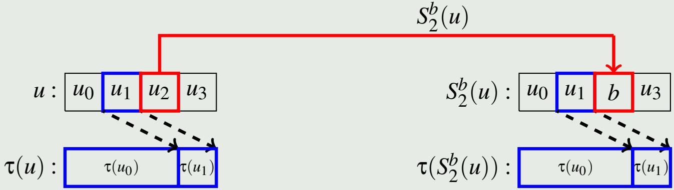

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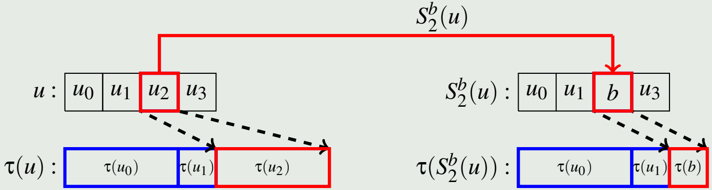

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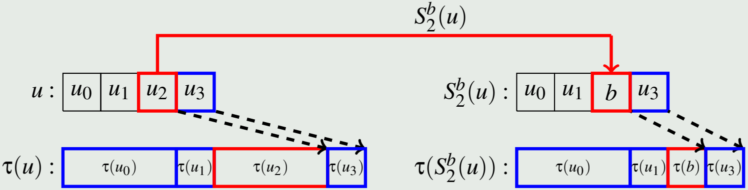

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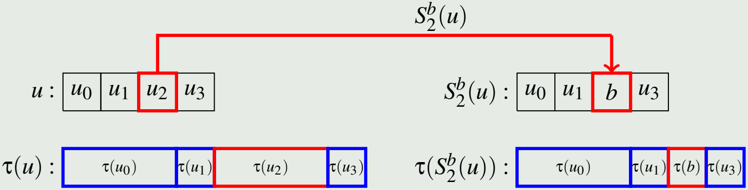

$$D_{n+p} \circ \cdots \circ D_{n+m+1} \circ S_{n+m}^{\tau(b)_m} \circ \cdots \circ S_{n+1}^{\tau(b)_1} \circ S_n^{\tau(b)_0}(\tau(u)) = \tau(S_2^b(u)).$$

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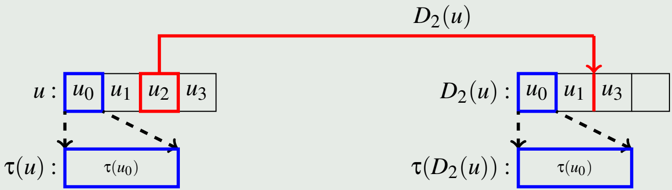

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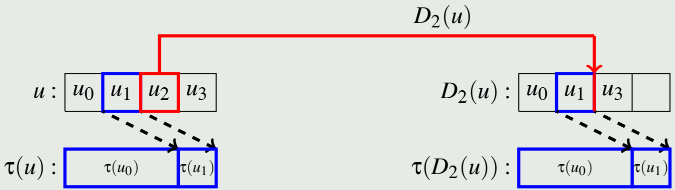

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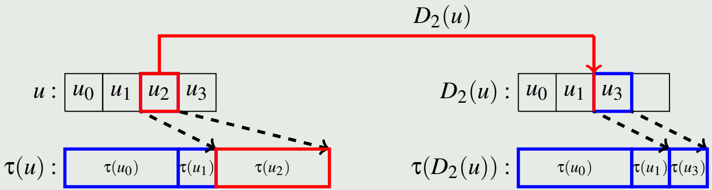

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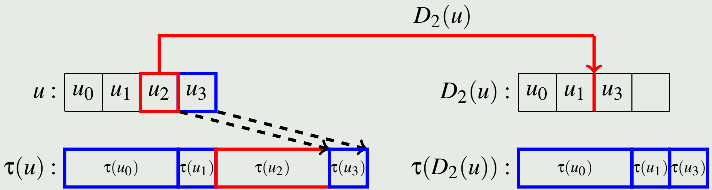

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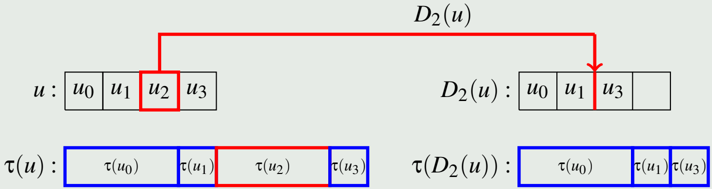

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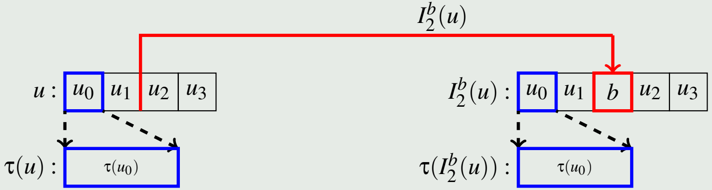

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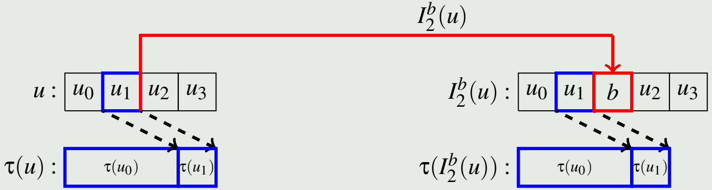

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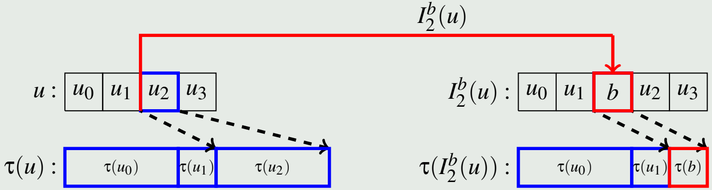

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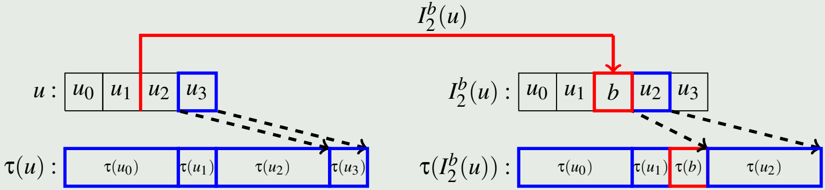

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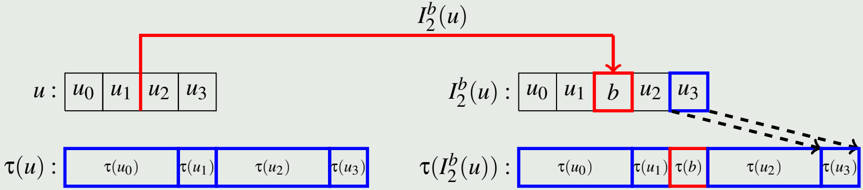

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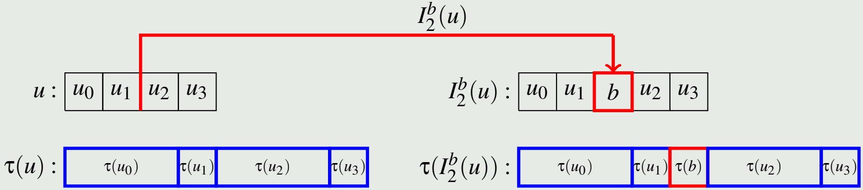

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Conclusion

- A metric which induces a non trivial topology.
- The shift map equal to the identity over this space.
- This topology turns out to be a suitable playground for the study of the dynamical behavior of CA and substitutions.
- This construction was made only by changing the Hamming distance with the edit distance.

Definitions

For a distance d over $A^* \times A^*$, we define the ***centred pseudo-metric*** *denoted by $\mathfrak{C}_d$ as follows :*

- $\mathfrak{C}_d(x,y) = \limsup_{l\to\infty} \dfrac{d(x_{[\![0,l[\![}, y_{[\![0,l[\![})}{max_{u,v\in A^l} d(u,v)}, \ \forall x,y \in A^{\mathbb{N}}.$

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Questions

A relevant questions is now the following :

- Which properties of distance d make CA or substitutions well-defined in the corresponding pseudo-metrics ?
- How other classic objects of symbolic dynamics behave ? (Minimal sub-shifts, Toeplitz, Sturmian, billiards ...)
- Can we give another version of the Curtis-Hedlund-Lyndon theorem with respect to centred and sliding pseudo-metrics ?

François Blanchard, Enrico Formenti, and Petr Kůrka, *Cellular automata in the Cantor, Besicovitch, and Weyl topological spaces*, Complex Systems **11** (1997), 107–123.

Gianpiero Cattaneo, Enrico Formenti, Luciano Margara, and Jacques Mazoyer, *A shift-invariant metric on s zz inducing a non-trivial topology*, International Symposium on Mathematical Foundations of Computer Science, Springer, 1997, pp. 179–188.

Gustav A Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Mathematical systems theory **3** (1969), no. 4, 320–375.

Vladimir I Levenshtein, *Binary codes capable of correcting deletions, insertions, and reversals*, Soviet physics doklady, vol. 10, 1966, pp. 707–710.

Johannes Müller and Christoph Spandl, *A Curtis–Hedlund–Lyndon theorem for Besicovitch and Weyl spaces*, Theoretical Computer Science **410** (2009), no. 38-40, 3606–3615 (en).